On the Forcing Open Monophonic Number of a Graph

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Abstract:

Problem: Let S be a minimum total (connected) open monophonic set of G. A subset T of S is called a forcing subset for S if S is the unique minimum total (connected) open monophonic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing total (connected)open monophonic number of S denoted by $f_{tom}(S)$ ($f_{com}(S)$)is the cardinality of a minimum forcing subset of S. The forcing total (connected) open monophonic number of G denoted by $f_{tom}(G)$ ($f_{com}(G)$) is $f_{tom}(G)$ ($f_{com}(G)$) $= min\{f_{tom}(S)\}$ ($min\{f_{com}(S)\}$), where the minimum is taken over all minimum total open monophonic sets in G. **Findings:** We determine bounds for it and characterize graphs which realize these bounds. Forcing total (connected) open monophonic number of certain standard graphs are found. Also, we proved that following results: (i) for any positive integers a, n with $0 \leq a \leq n-4$, there exists a connected graph G of order n such that $f_{tom}(G) = 0$ or 1, or 2 and $om_t(G) = a$. (ii) for any integer b with $b \geq 6$, there exists a connected graph G such that $f_{com}(G) =$ 0 or 1 and $om_c(G) = b$.
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1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops or multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to Harary [4]. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An u-v path of length $d(u,v)$ is called an *u-vgeodesic*. It is known that this distance is a metric on the vertex set $V(G)$. For any vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad G* and the maximum eccentricity is its *diameter*, *diam G* of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices which are adjacent with v. The vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. For a cutvertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v. A *geodetic* set of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the cardinality of a minimum geodetic set. A vertex x is said to *lie* on a *u-vgeodesic P* if x is a vertex of P and x is called an *internal vertex* of P if $x \neq u, v$. A set S of vertices of a connected graph G is an *opengeodeticset*of *G*if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of a *x-y*geodesic for some $x, y \in S$. "An open geodetic set of minimum cardinality is a minimum open geodetic set and this cardinality is the *open geodetic numberog*(G). It is clear that every open geodetic set is a geodetic set so that$g(G) \leq og(G)$. The geodetic number of a graph was introduced and studied in [**1, 2**]. The open geodetic number of a graph was introduced and studied in [3**, 5, 7**] in the name open geodomination in graphs. A chord of a path $u_1, u_2, \ldots, u_n$ in G is an edge u_iu_j with $j \geq i + 2$.

2. Methodology

For two vertices u and v in a connected graph G, a *u-v* path is called a *monophonic path* if it contains no chords. A *monophonic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a monophonic path joining some pair of vertices in S. The *monophonic numberm*(G) of G is the cardinality of a minimum monophonic set.

A set S of vertices in a connected graph G is an *open monophonic set* if for each vertex v in G, either v is an extreme vertex of G and $v \in S$, or v is an internal vertex of an *x-ymonophonic path*for some $x, y \in S$. An open monophonic set of minimum cardinality is a *minimum open monophonic set* and this cardinality is the *open monophonic numberom*(G) of G. This concept was introduced and studied in [9].A connected open monophonic set of G is an open monophonic set$S$such that the subgraph $<S>$ induced by S is connected. The minimum cardinality of a connected open monophonic set of G is the connected open monophonic number, $om_c(G)$. This concept was introduced and studied in [8]. A *total open monophonic set* of a graph G is an open monophonic set$S$ such that the subgraph $<S>$ induced by S contains no isolated vertices. The minimum cardinality of a total open monophonic set of G is the *total open monophonic number* of G and is denoted by $om_t(G)$ of G.This concept was introduced and studied in [10]. Let S be a minimum open monophonic set of G. A subset T of S is called a for cingsubset for S if S is the unique minimum open monophonic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing open monophonic number of S, denoted by $f_{om}(S)$, is the cardinality of a minimum forcing subset of S. The forcing open monophonic number of G, denoted by $f_{om}(G)$, is $f_{om}(G) = min(f_{om}(S))$, where the minimum is taken over all minimum open monophonic sets in G. This concept was introduced and studied in [11]. A set S of vertices of G is a restrained open monophonic set if either $S = V$ or S is an open monophonic set and the sub graph induced by $V - S$ has no isolated vertex. A restrained open monophonic set of minimum cardinality is a minimum restrained open monophonic set and this cardinality is the restrained open monophonic number of G, denoted by $om_r(G)$. This concept was introduced and studied in [12].A total open monophonic set S of vertices in a connected graph G is a minimal total open monophonic set if no proper subset of S is a total open monophonic set of G. The upper total open monophonic number $om+_t(G)$ is the maximum cardinality of a minimal total open monophonic set of G. This concept was introduced and studied in [13]. A connected open monophonic set S of vertices in a connected graph G is a minimal connected open monophonic set if no proper subset of S is an open monophonic set of G. The upper connected open monophonic number $om+_c(G)$ is the maximum cardinality of a minimal connected open monophonic set of G". This concept was introduced and studied in [14].

The following theorem are used in the sequel.

Theorem 2.1 [10]: Every total open monophonic set of a connected graph *G* contains all its extreme vertices and support vertices. If the set of all extreme vertices and support vertices form a total open monophonic set of *G*, then it is the unique minimum total open monophonic set of *G*.

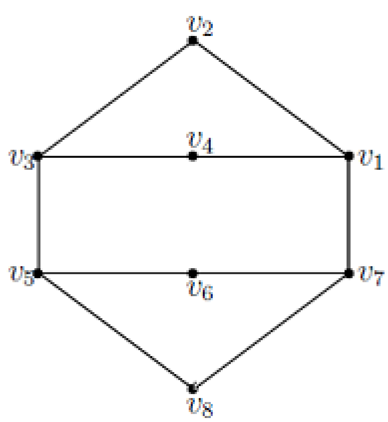
Theorem 2.2 [11]: Let *G* be a connected graph. Then
(i) $f_{om}(G) = 0$ if and only if *G* has a unique minimum open monophonic set.
(ii) $f_{om}(G) = 1$ if and only if *G* has at least two minimum open monophonic sets, one of which is a unique minimum total open monophonic set containing one of its elements.
(iii) $f_{om}(G) = om(G)$ if and only if no minimum open monophonic set of *G* is the unique minimum open monophonic set containing any of its proper subsets.

Theorem 2.3 [8] Every extreme vertex of a connected graph *G* belongs to each connected open monophonic set of a graph *G*. In particular, every end vertex of *G* belongs to each connected open monophonic set of *G*.

3. Results and Discussion

Definition 1 Let *G* be a connected graph and *S* a minimum total open monophonic set of *G*. A subset *T* of *S* is called a forcing subset for *S* if *S* is the unique minimum total open monophonic set containing *T*. A forcing subset for *S* of minimum cardinality is a minimum forcing subset of *S*. The forcing total open monophonic number of *S* denoted by $f_{tom}(S)$ is the cardinality of a minimum forcing subset of *S*. The forcing total open monophonic number of *G* denoted by $f_{tom}(G)$ is $f_{tom}(G) = min\{f_{tom}(S)\}$, where the minimum is taken over all minimum total open monophonic sets in *G*.

Example 2 For the graph *G* in Figure 1, $S_1 = \{v_2, v_3, v_7, v_8\}$, $S_2 = \{v_1, v_2, v_5, v_8\}$, $S_3 = \{v_1, v_4, v_5, v_6\}$, $S_4 = \{v_3, v_4, v_6, v_7\}$ and $S_5 = \{v_1, v_3, v_5, v_7\}$ are the minimum total open monophonic sets of G, so that $om_t(G) = 4$.



G
Figure 1

No single vertex is a forcing subset for any of the S_i ($1 \leq i \leq 5$). Since $\{v_2, v_3\}$is a forcing subset for $S_1 f_{tom}(S_1) = 2$ and so $f_{tom}(G) = 2$. The next theorem followsimmediately from the definitions of total open monophonic number and forcing total open monophonic number of a connected graph G

Theorem 3: For every connected graph G, $0 \leq f_{tom}(G) \leq om_t(G) \leq n$.

Remark 4: For the complete graph $G = K_n (n \geq 2)$, the set of all vertices of $K_n$is the unique minimum total open monophonic set so that $om_t(G) = n$ and $f_{tom}(G) = 0$.For the complete bipartite graph $K_{r,s}(3 \leq r \leq s)$, $om(K_{r,s}) = 4 = f_{tom}(K_{r,s})$. Also, all the inequalities in Theorem 2.2 can be strict.For the graph G given in Figure 2.1, $om_t(G) = 4$, $f_{tom}(G) = 2$ and $n = 8$. Hence$0 < f_{tom}(G) < om_t(G) < n$.

Theorem 5: Let G be a connected graph. Then
(i) $f_{tom}(G) = 0$ if and only if *G* has a unique minimum total open monophonic set.
(ii) $f_{tom}(G) = 1$ if and only if *G* has at least two minimum total open monophonic sets, oneof which is a unique minimum total open monophonic set containing one of its elements.
(iii) $f_{tom}(G) = om_t(G)$ if and only if no minimum total open monophonic set of G is the uniqueminimum total open monophonic set containing any of its proper subsets.

Proof: The proof of this theorem similarly to **Theorem 1.2**

Theorem 6: For the graph path P_n, $(n \geq 5)$, $f_{tom}(G) = 0$.
Proof: Let $P_n : v_1 v_2 \ldots v_n$ be the path of order n. It is clear that $\{v_1, v_2, v_{n-1}, v_n\}$ is unique totol open monophonic set and hence by Theorem 2.4, $f_{tom}(G) = 0$.

Theorem 7. For any cycle G = Cn($n \geq 4$), $f_{tom}(G) = \begin{cases} 4 & n \geq 5 \\ 0 & n = 4 \end{cases}$

Proof. Let $G = C_n (n \geq 4)$ be the cycle $C_n : v_1, v_2, v_3, \ldots, v_n, v_1$.

For n = 4, $S = \{v_1, v_2, v_3, v_4\}$ is the unique open monophonic set of G so that by Theorem 2.4(i), $f_{tom}(G) = 0$.

Let G = Cn($n \geq 5$). Then any set containing 4 consecutive vertices or two pair of consecutive vertices of G is a minimum open monophonic set so that $om_t(G) = 4$. Since $n \geq 5$,it is clear that no 3-element subset of any minimum total open monophonic setS is a forcing subset for S. Hence $f_{tom}(G) = 4$.

Theorem 8. For any wheel $W_n = K_1 + C_{n-1} (n \geq 4)$, $f_{tom}(w_n) = \begin{cases} 4 & n \geq 6 \\ 0 & n = 4, 5 \end{cases}$

Proof. Let $C_{n-1} : v_1, v_2, \ldots, v_{n-1} v_1$. Let $K_1 = \{x\}$. Then $Wn = C_{n-1} + K_1$.

For n = 4, W_4 is the complete graph K_4 so that $f_{tom}(G) = 0$.

For n = 5, the set $S = \{v_1, v_2, v_3, v_4\}$ is the unique open monophonicset of G, then by Theorem 2.4(i), $f_{tom}(G) = 0$.

Let $W_n = K_1+C_{n-1}(n \geq 8)$. Then any set containing 4 consecutive vertices or two pair of consecutive vertices of W_n is a minimum open monophonic set so that $om_t(W_n) = 4$. Since $n \geq 5$, it is clear that no 3-element subset of any minimum total open monophonic set S is a forcing subset for S. Hence $f_{tom}(W_n) = 4$.

Theorem 9. For the complete bipartite graph G = Kr,s$(2 \leq r \leq s)$,

$$f_{tom}(G) = \begin{cases} 0 & if\ r = s = 2 \\ 2 & if\ 2 = r < s \\ 4 & if\ 3 \leq r \leq s \end{cases}$$

Proof. Case 1. r = s = 2. Then G is the cycle C_4 and by Theorem 2.6, $f_{tom}(G) = 0$.

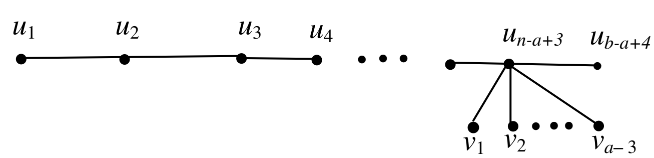
Case 2. 2 = r < s. Let U and W be the partite sets of $K_{r,s}$ with|U | = r and |S| = s. It is clear that the minimum total open monophonic sets Sof G are obtained by choosing the two elements of U and any two elementsfrom W. It is easily verified that no single vertex subset of any of thesesets is a forcing subset for S. Also, it is clear that any 2-element subset ofW is a forcing subset for one of these minimum total open monophonic sets S of
G. Hence $f_{tom}(G) = 2$.

Case 3. $3 \leq r \leq s$. Then any minimum total open monophonic set S is gotby taking any two elements from U and any two elements from W. It iseasily verified that no 3-element subset of any minimum total open monophonicsets S of G is a forcing set for S. Hence it follows that $f_{tom}(G) = 4$.

Problem 10 Characterize graphs G for which $om_t(G) = f_{tom}(G)$.

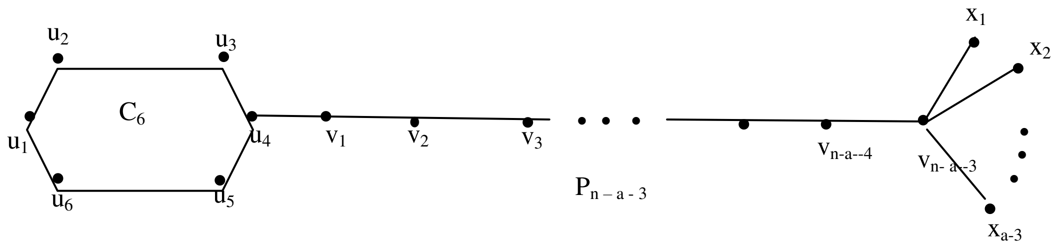
Theorem 11 For any positive integers a, n with $0 \leq a \leq n-4$, there exists a connected graph G of order n such that $f_{tom}(G) = 0$ or 1, or 2 and $om_t(G) = a$.

Proof. Case 1. $f_{tom}(G) = 0$. Let $P_{n-a+4} : u_1, u_2, u_3, u_4, \ldots, u_{n-a+4}$ be a path of order $n-a+4$. Let G be the graph of order n in Figure 2, obtained from P_{n-a+4} by adding the new vertices $v_1, v_2, \ldots, v_{a-4}$ and joiningeach v_i $(1 \leq i \leq a-4)$ with u_1. Let $S = \{v_1, v_2, v_3, \ldots, v_{a-4}, u_1, u_2, u_{n-a+3}, u_{n-a+4}\}$. Then $S$is the unique minimum total open monophonic set and so by Theorems 1.1 and 2.4(i), it follows that $om_t(G) = a$ and$f_{tom}(G) = 0$.



G
Figure 2

Case 2.$f_{tom}(G)$= 1. Let $C_6 : u_1, u_2, u_3, u_4, u_5, u_6, u_1$ be a cycle of order 6. Let P_{n-a-3}: $v_1, v_2, \ldots, v_{n-a-3}$ be the path of order $n-a-3$. Let H be the graph obtained from C_6 and P_{n-a-3} by joining the vertex u_4 with v_1. Now, by adding the new vertices $x_1, x_2, \ldots, x_{a-3}$ with v_{n-a-3} we obtained the graph G in Figure 3 of order$n$.



G
Figure 3

Let $S = \{x_1, x_2, x_3, \ldots, x_{a\text{-}3}, v_{n\text{-}a\text{-}3}\}$. By Theorem 2.1, every total open monophonic set of G contains S. It is clear that S is not a total open monophonic set of G. Also, it is clear that for any $x \notin S$, $S \cup \{x\}$ is not a total open monophonic set of G. Now, it is easily verified that $S_1 = S \cup \{u_1, u_2\}$ and $S_2 = S \cup \{u_1, u_6\}$ are the only two total open monophonic sets of G, $om_t(G) = a$ and by Theorem 2.4(ii), $f_{tom}(G) = 1$.

Case 3. $f_{tom}(G) = 2$. Let $C_{n\text{-}a+3}$: $v_1, v_2, v_3, \ldots, v_{n\text{-}a+3}, v_1$be a cycle of order $n - a + 3$. Let G be thegraph in Figure 4, obtained from $Cn\text{-}a+3$ by adding the new vertices $u_1, u_2, \ldots, u_{a\text{-}3}$ andjoining each $u_i (1 \leq i \leq a - 3)$ with v_1.

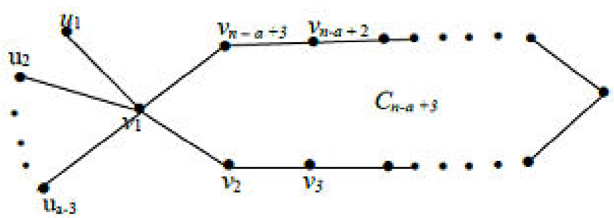


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Figure 4

Let $S' = \{u_1, u_2, \ldots, u_{a\text{-}4}, v_1\}$. By Theorem 1.1, every total open monophonic set of Gcontains S. It is clear that $S$is not a total open monophonic set of G. Also, it is clear thatfor any $x \notin S$, $S \cup \{x\}$ is not a total open monophonic set of G. Now, it is easily verified that$S$i $= S \cup \{vi, v_{j+1}\}$, where vi is non adjacent to v_2 and $v_{n\text{-}a+4}$are minimum total open monophonic sets of G so that $om_t(G) = a$. Also, it is observed that no singleton set cannot be a forcing subset of S_i. Also, it is clear that the subset $\{v_i, v_{j+1}\}$ contained only in S_i and hence $f_{tom}(G) = 2$.

Definition 12 Let G be a connected graph and S a minimum connected open monophonic set of G. A subset T of S is called a forcing subset for S if S is the unique minimum connected open monophonic set containing T. A forcing subset for S of minimum cardinality is a minimum forcing subset of S. The forcing connected open monophonic number of S denoted by $f_{com}(S)$ is the cardinality of a minimum forcing subset of S. The forcing connected open monophonic number of G denoted by $f_{com}(G)$ is $f_{com}(G) = min\{f_{com}(S)\}$, where theminimum is taken over all minimum connected open monophonic sets in G.

Example 13 For the graph G in Figure 5, $S_1 = \{v_1, v_2, v_3, v_6\}$ and $S_2 = \{v_3, v_4, v_5, v_6\}$, are the minimum connected open monophonic sets of G, so that $om_c(G) = 4$. It is clear that
$f_{com}(S_1) = 1$ and $f_{com}(S_2) = 1$. Hence $f_{com}(G) = 1$.

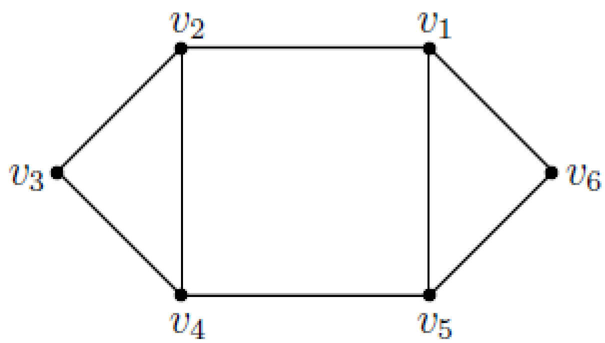

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Figure 5

The next theorem follows immediately from the definitions of connected open monophonic number and forcing connected open monophonic number of a connected graph G.

Theorem 14: For every connected graph G, $0 \leq f_{com}(G) \leq om_c(G) \leq n$.

Remark 3.3: For the complete graph $G = K_n (n \geq 2)$, the set of all vertices of $K_n$is the unique minimum total open monophonic set so that $om_c(G) = n$ and $f_{com}(G) = 0$. For the complete bipartite graph $K_{r,s}(3 \leq r \leq s)$, $om_c(K_{r,s}) = 4 = f_{com}(K_{r,s})$. Also, all the inequalities in Theorem 14 can be strict. For the graph G given in Figure , $om_c(G) = 4$, $f_{com}(G) = 1$ and n = 6. Hence $0 < f_{com}(G) < om_c(G) < n$.

The following theorem gives necessary and sufficient conditions for the forcing connected open monophonic number $f_{com}(G)$ of a graph to be 0 or 1 or $om_c(G)$ for a graph G.

Theorem 15: Let G be a connected graph. Then
(i) $f_{com}(G) = 0$ if and only if G has a unique minimum connected open monophonic set.
(ii) $f_{com}(G) = 1$ if and only if G has at least two minimum connected open monophonic sets, one of which is a unique minimum connected open monophonic set containing one of its elements.
(iii) $f_{com}(G) = om_c(G)$ if and only if no minimum connected open monophonic set of G is the unique minimum connected open monophonic set containing any of its proper subsets.

Proof: The proof of this theorem similarly to **Theorem 2.2**

Definition 16 A vertex v of a connected graph G is said to be a connected open monophonic vertex of G if v belongs to every minimum connected open monophonic set of G.

Example 17 For the graph G given in Figure 3.2, $S_1 = \{u, v, x, y\}$, $S_2 = \{u, v, x, z\}$ and $S_3 = \{u, v, y, z\}$ are the minimum connected open monophonic sets of G so that u and v are the connected open monophonic vertices of G.

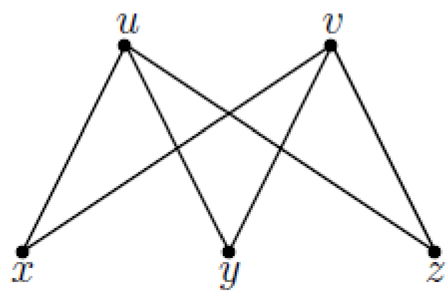

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Figure 3.2

Theorem 18: For the complete graph K_n, $(n \geq 5)$, $f_{com}(G) = 0$.

Proof: For G = Kn, it follows from Theorem 1.3 that the set of all vertices of G isthe unique connected open monophonic set. Hence it follows from Theorem 3.4 (i) that$fcom(G) = 0$.
If G is a non-trivial tree, then by Theorem 1.3, the set of all vertices of $G$is the unique minimum connected open monophonic set of G and so by Theorem 3.4(i), $fcom(G) = 0$.

We leave the following problem as an open question.

Problem 19 Characterize graphs G for which fcom(G) = $om_c(G)$.

Theorem 20 Let G be a connected graph with $om_c(G) = 2$. Then $f_{com}(G) = 0$.

Proof. Let $S = \{u, v\}$ be a om_c-set of G. Since the subgraph induced by S is connected it follows that $G = K_2$. Hence $f_{com}(G) = 0$.

In view of Theorem 14, we have the following realization theorem.

Theorem 21 For any integer$b$ with $b \geq 6$, thereexists a connected graph G such that $f_{com}(G) =$ 0 or 1 and $om_c(G) = b$.

Proof. We prove this theorem by considering two cases.

Case 1. $f_{com}(G) = 0$. Let $G = K_b$. Then, by Theorems 1.3 and 3.4, we have $f_{com}(G) = 0$and $om_c(G) = b$.

Case 2. $f_{com}(G) = 1$. Let G be the graph in Figure 3.3, obtained from $C_4 : v_1, v_2, v_3, v_4, v_1$byfirst adding $b - 4$ new vertices $u_1, u_2, u_3, \ldots, u_{b-4}$ to C_4 and joining each $u_i (1 \leq i \leq b-4)$with the vertex v_1 of C_4; and also adding the new vertex v_5 and joining theedges v_5v_2 andv_5v_4.

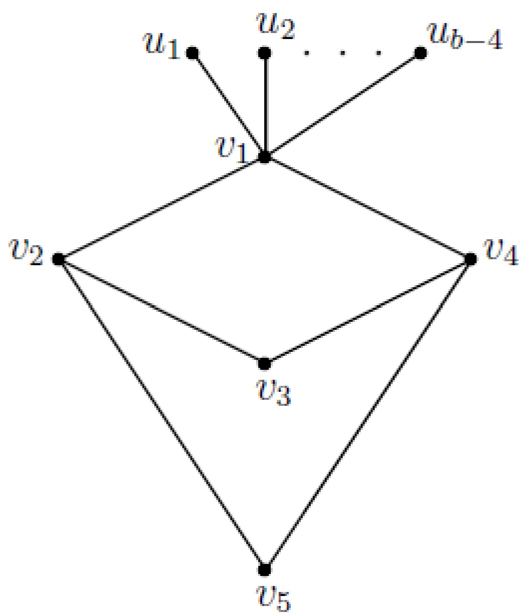

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Figure 3.3

Let $S' = \{u_1, u_2, \ldots, u_{b-4}, v_1\}$. By Theorems 1.3, every connectedopen monophonic set of*G* contains S'. Then *G* contains exactly two minimum connectedopen monophonic sets namely $S_1 = S' \cup \{v_2, v_3, v_4\}$ and $S_2 = S' \cup \{v_2, v_4, v_5\}$. Thus$om_c(G) = |S'| + 4 = b$. Since$S_1$ is the unique minimum connected open monophonic setcontaining v_3, it follows from Theorem 3.4 (ii) that f_{com}G) = 1 = a.

Implementation: The concept of forcing open monophonic sets has interesting application in Channel Assignment Problem in FM radio technologies. The forcing monophonic matrix is used to discuss different aspects of standard molecular graphs corresponding molecules arising in special situations of molecular problems in theoretical Chemistry,

Conclusion: In this paper, we have introduced two new parameters such as forcing total open monophonic number and forcing connected open monophonic number. Also, we studied about the bounds, existence and realization of these parameters.

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