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Optimum Future Premiums and Discounts Using Utility Functions

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Abstract: In this paper we try to determine discounted future optimum premium values for exponential and quadratic utility function using optimal control theory such as Hamiltonian- Jacobi Bellman (HJB) equation. We optimize level discounted future premiums for the annuity using some continuous and discrete analogues of continuous loss distributions. Comparisons are made to level premiums for these distributions using arbitrary values of force of interest.
Keywords: Loss distribution, optimal control theory, utility function, premiums and discount.

1. Introduction

Premium and discount in futures is an important concept to understand as it plays a significant role in the global financial system. It affects the way traders buy and sell contracts over a period of time, making it essential to have knowledge about this component when trading commodities or other derivatives. Additionally, understanding how premiums and discounts work intern help investors to manage their risk better by avoiding unwarranted risks that may be associated with changes in market rates or health issues. With careful, one can make sound decisions even during times when systems (market or health) appear volatile or unpredictable.

Stracke and Heinen (2006) brought the impact on an insured and the relationship between insurance and influenza pandemic. Feng and Garrido (2011) gave actuarial applications to epidemic model and explained financial arrangements to cover the expenses resulting from the medical treatments of infectious diseases. They used SIR (Susceptible-Infected-Removal) model. Ishii, et al. (2015) used simple stochastic epidemic model to estimate epidemic risks for an insurer. Claude, et al. (2017) applied actuarial methods to develop the insurance plan for protecting against an epidemic risk. They considered an extended SIR epidemic in which the removal and infection rates may depend on the number of registered removals. The premiums to be paid during an epidemic outbreak are measured through the expected susceptibility time.To see the actuarial applications of discrete analogues of continuous loss distributions, Mallappa and Talawar (2021) have used a simple deterministic epidemic model for numerical illustration to calculate future premiums and annuities.

Mallappa and Talawar (2020) determined the maximum premium for the insured to pay by considering different forms of utility functions, assuming the loss random variable to follow different forms of discrete analogues continuous distributions. For discrete analogues of continuous loss distributions see, Chakraborty, S. (2015).In the present paper we have used stochastic optimal control theory to calculate discounted optimum future premiums for exponential and quadratic utility functions and then considered the continuous and discrete analogues of continuous loss distributions to calculate discounted optimum future premiums using simple compartmental model.

In section 2, we deal with utility functions using stochastic optimal control to calculate discounted future premiums and claims for different continuous and discrete analogues of continuous loss distributions. Section 3 deals with calculation of future premiums and provides numerical illustrations about the calculation of future premiums and future claims. Section 4 gives conclusion of the results.

2. Stochastic optimal control theory to discounted future premiums

We assume, as in Feng and Garrido (2011) that, the infection disease protection plan works in a simple annuity fashion. Premiums are collected continuously as long as the insured person remains susceptible and medical expenses are continuously reimbursed to infected policy holder during the period of treatments.

Following the international actuarial notation, the actuarial present value(APV) of premium payments from an insured person for the whole epidemic is denoted by ${\bar{a}^s}_0$ with the superscript indicating payments form susceptible class S and the APV of benefit payments from the insurer is denoted by ${\bar{a}^i}_0$ with superscript indicating payment to infected class I.

On the revenue side, the total discounted future premium is

$${\bar{a}^s}_0 = \int_0^\infty e^{-\delta t} s(t)dt$$

(1)

On the debit side of the insurance product, the total discounted future claim is given by

$${\bar{a}^i}_0 = \int_0^\infty e^{-\delta t} i(t)dt$$

(2)

Where δ is the force of interest.

We present the computation of optimum discounted future premium values using exponential and quadratic utility function. We optimize the discounted future premium subject to the condition

Exponential utility function i.e., $t = -\lambda e^{-\lambda w}, w \geq 0$

(3)

Quadratic utility function i.e., $t = -(a-w)^2, a \geq w$

(4)

We use Hamiltonian Jacobi Bellman (HJB) equation to find the optimum future premium and it is given in the following form

$$H = \int_0^\infty e^{-\delta t} s(t)dt + \lambda t$$

(5)

Where λ is Hamiltonian-Jacobi Bellman constant and it can be obtained using the following condition

$$\frac{\partial H}{\partial t} = 0$$

$$\Rightarrow \int_0^\infty \frac{\partial}{\partial t}\left(e^{-\delta t}s(t)\right)dt + \lambda$$

$$= \left[e^{-\delta t}s(t)\right]_0^\infty + \lambda$$

$$= -s(0) + \lambda$$

$$\hat{\lambda} = s(0)$$

(6)

And $\frac{\partial^2 H}{\partial t^2} = \int_0^\infty \frac{\partial^2}{\partial t^2}\left(e^{-\delta t}s(t)\right)dt$

$$\frac{\partial^2 H}{\partial t^2} = -s'(0) + \delta s(0) < 0$$

Thus, the solution for HJB equation given in the following simple equation

$$P_{optF} = \int_0^\infty e^{-\delta t} s(t)dt + s(0)\, t$$

(7)

Where, P_{optF} is the discounted optimum future premium value

3. Calculation and Numerical Illustration:

In the following tables we give discounted future premium values for discrete analogues of continuous loss distributions subject to the conditions of exponential and quadratic utility functionsassuming $a = 105, w = 104$ and arbitrary force of interest $\delta = 0.02, 0.04, 0.06, 0.08$ and 0.10.

3.1 Discounted optimum future premium values under exponential utility function.

Table 1 and Table 2 give the discounted optimum future premium values under exponential utility function for some continuous and discrete analogues of continuous loss distributions respectively.

Table 1:Discounted optimum future premium values underexponential utility function for some continuous distributions.

Probability Distribution	Force of interest ($\boldsymbol{\delta}$)	Discounted optimum future premium values for exponential utility function($\boldsymbol{\lambda = 2}$)
Exponential distribution	0.02 0.04 0.06 0.08 0.10	0.9803 0.9615 0.9433 0.9256 0.9090 $\}$ $for \theta = 1,$
Gamma distribution (α, θ)	0.02 0.04 0.06 0.08 0.10	0.9611 0.9245 0.8899 0.8573 0.8265 $\}$ $for \theta = 1, \alpha = 2$
Weibull distribution (α)	0.02 0.04 0.06 0.08 0.10	0.9824 0.9653 0.9485 0.9321 0.9161 $\}$ $for \alpha = 2$
Beta distribution second kind (α, β)	0.02 0.04 0.06 0.08 0.10	0.9637 0.9319 0.9032 0.8768 0.8524 $\}$ $for \alpha = 2, \beta = 2$
Rayleigh distribution (σ^2)	0.02 0.04 0.06 0.08 0.10	0.9514 0.9058 0.8629 0.8227 0.7848 $\}$ $for \sigma = 2$

From the above, we notice the following

- Discounted optimum future premium values decrease as force of interest increases

- Weibull ($\alpha = 2$) and exponential ($\theta = 1$) distributions give slightly higher future premium values as compared to other continuous distributions.
- We observe that, there is no much differences in discounted future premium values and maximum premium values.
- Also we observe here, future premium values are slight less if we assume loss as discrete analogues of some arbitrary continuous distributions under this utility function

Table 2: Discounted optimum future premium values underexponential utility function for some discretized continuous loss distributions.

Probability Distribution	**Force of interest (δ)**	**Discounted optimum future premium values for Exponential Utility function($\lambda = 2$)**
Discrete Exponential distribution	0.02 0.04 0.06 0.08 0.10	0.9969 0.9939 0.9909 0.9881 0.9853 $for \theta = 2$
Discrete Gamma distribution	0.02 0.04 0.06 0.08 0.10	0.9938 0.9878 0.9820 0.9763 0.9708 $for m = 2, \theta = 2$
Discrete Weibull distribution	0.02 0.04 0.06 0.08 0.10	0.9923 0.9848 0.9775 0.9704 0.9634 $for \beta = 2$
Discrete Rayleigh distribution	0.02 0.04 0.06 0.08 0.10	0.9973 0.9946 0.9921 0.9895 0.9870 $for \sigma^2 = 0.25$
Discrete Pareto distribution	0.02 0.04 0.06 0.08 0.10	0.9719 0.9647 0.9577 0.9511 0.9448 $for \beta = 2$

Discrete Burr distribution	0.02 0.04 0.06 0.08 0.10	0.9933 0.9875 0.9819 0.9765 0.9712 $\Bigg\}$ $for\alpha = 2, \beta = 2$

From the Table 2, we observe the following

- Discounted future premium values decrease as force of interest increases
- Future premium values slight high if we assume loss random variable as discrete analogues of continuous Rayleigh distribution with parameter$\sigma = 0.25$.
- Also we observe that, future premium values are little high if we assume loss random variable as discrete analogues of continuous distribution under this utility function

3.2 Discounted optimum future premium values under quadratic utility function.

Table 3 and Table 4 give the discounted optimum future premium values under quadratic utility function for some continuous and discrete analogues of continuous loss distributions respectively.

Table 3: Discounted optimum future premium values under quadratic utility function for some continuous distributions.

Probability Distribution	Force of interest ($\boldsymbol{\delta}$)	Discounted optimum future premium values for Quadratic utility function
Exponential distribution	0.02 0.04 0.06 0.08 0.10	0.4615 0.4259 0.3928 0.3620 0.3333 $\Bigg\}$ $for\theta = 0.5,$
Gamma distribution (α, θ)	0.02 0.04 0.06 0.08 0.10	0.9245 0.8573 0.7971 0.7431 0.6944 $\Bigg\}$ $for\theta = 0.5, \alpha = 2$
Weibull distribution (α)	0.02 0.04 0.06 0.08 0.10	0.9823 0.9649 0.9480 0.9313 0.9150 $\Bigg\}$ $for\alpha = 3$
Beta distribution second kind (α, β)	0.02 0.04 0.06 0.08 0.10	0.9408 0.8950 0.8560 0.8216 0.7908 $\Bigg\}$ $for\alpha = 2, \beta = 1.5$
Rayleigh distribution (σ^2)	0.02 0.04 0.06 0.08 0.10	0.9753 0.9514 0.9282 0.9053 0.8840 $\Bigg\}$ $for\sigma = 1$

From the above, we notice the following

- We observe similar pattern as in Table 1 and Table 2 that, discounted future premiums decrease as force of interest increases.
- Future premium values are very low if we assume loss as exponential distribution with parameter$\theta = 0.5$. But future premium value increases with increase in the value of θ.
- We can observe that, there is no much difference in discounted future premium values and maximum premium values except in the case of exponential distribution with parameter $\theta = 0.5$

Table 4: Discounted optimum future premium values under quadratic utility function for some discretized distributions.

Probability Distribution	Force of interest ($\boldsymbol{\delta}$)	Discounted optimum future premium values for Quadratic utility function
Discrete Exponential distribution	0.02 0.04 0.06 0.08 0.10	0.1322 0.1292 0.1263 0.1234 0.1206 $for \theta = 2$
Discrete Weibull distribution	0.02 0.04 0.06 0.08 0.10	0.3602 0.3527 0.3454 0.3382 0.3312 $for \beta = 2$
Discrete Rayleigh distribution	0.02 0.04 0.06 0.08 0.10	0.1326 0.1300 0.1274 0.1249 0.1224 $for \sigma^2 = 0.25$
Discrete Pareto distribution	0.02 0.04 0.06 0.08 0.10	0.2219 0.2147 0.2077 0.2011 0.1948 $for \beta = 2$
Discrete Burr distribution	0.02 0.04 0.06 0.08 0.10	0.2433 0.2357 0.2319 0.2265 0.2212 $for \alpha = 2, \beta = 2$

From the abovetable, we notice that, discounted future premium values are very low if we assume loss as discrete analogues of some continuous distributions under quadratic utility function.

Table 5: Level discounted future premiums for the annuity using exponential utility function for some continuous distributions.

$\bar{a}_0^i / \bar{a}_0^s$	Exponential	Gamma	Weibull	Beta	Rayleigh
Exponential		0.9804 0.9511 0.9434 0.9363 0.8962	1.0011 1.0040 1.0055 1.0169 1.0078	0.9831 0.9692 0.9574 0.9576 0.9366	0.9705 0.9421 0.9137 0.8665 0.8634
Gamma	1.0200 1.0514 1.0600 1.0680 1.1133		1.0211 1.0555 1.0659 1.0861 1.1220	1.0027 1.0190 1.0148 1.0227 1.0427	0.9899 0.9905 0.9685 0.9468 0.9612
Weibull	0.9989 0.9961 0.9945 0.9834 0.9922	0.9793 0.9474 0.9382 0.9207 0.8913		0.9820 0.9654 0.9521 0.9417 0.9294	0.9694 0.9384 0.9087 0.8718 0.8567
Beta	1.0172 1.0318 1.0445 1.0443 1.0677	0.9973 0.9813 0.9854 0.9778 0.9590	1.0184 1.0358 1.0503 1.0619 1.0760		0.9872 0.9720 0.9544 0.9258 0.9218
Rayleigh	1.0304 1.0615 1.0944 1.1280 1.1583	1.0102 1.0096 1.0325 1.0562 1.0404	1.0315 1.0657 1.1005 1.1471 1.1673	1.0129 1.0288 1.0478 1.0802 1.0849	

Table 5 gives the level optimum premiums for annuity assuming future optimum premium to be paid by the susceptible individual or future optimum claim to be made by the infected individual follow some continuous loss distributions under exponential utility function. If we assume exponential for future optimum premium and all other distributions for future optimum claim, the level of optimum less than unity for all distributions except Weibull distribution and in all the cases the values are decreasing with increase in rate of interest. If we assume Gamma for future optimum premium, then the level of optimum greater than unity for all distributions except Rayleigh distributionand in all the cases the values are decreasing with increase in rate of interest.If we assume Weibull for future optimum premium, the level of optimum less than unity for all distributions and in all the cases the values are decreasing with increase in rate of interest.If we assume Beta for future optimum premium, the level of optimum less than unity for Gamma and Rayleigh distributionsand in these cases the values are decreasing with increase in rate of interest and greater than unity for Beta and Weibull distributions and in these cases the values are increasing with increase in rate of interest. If we assume Rayleigh for future optimum premium, the level of optimum greater than unity for all distributions and in all the cases the values are increasing with increase in rate of interest.

Table 6: Level discounted future premiums for the annuity using exponential utility function for some discrete analogues of continuous distributions.

$\bar{a}_0^i / \bar{a}_0^s$	Discrete Exponential	Discrete Gamma	Discrete Weibull	Discrete Rayleigh	Discrete Pareto	Discrete Burr
Discrete Exponential		0.9969 0.9939 0.9900 0.9881 0.9853	0.9944 0.9908 0.9865 0.9821 0.9853	1.0004 1.0007 1.0002 1.0014 1.0017	0.9749 0.9706 0.9665 0.9626 0.9589	0.9964 0.9936 0.9909 0.9883 0.9856
Discrete Gamma	1.0031 1.0062 1.0101 1.0121 1.0149		0.9975 0.9970 0.9964 0.9940 0.9924	1.0035 1.0069 1.0103 1.0135 1.0167	0.9780 0.9766 0.9762 0.9742 0.9732	0.9995 0.9997 1.0009 1.0002 1.0003
Discrete Weibull	1.0056 1.0092 1.0137 1.0182 1.0227	1.0025 1.0030 1.0036 1.0061 1.0077		1.0061 1.0100 1.0139 1.0197 1.0245	0.9804 0.9796 0.9797 0.9801 0.9807	1.0020 1.0027 1.0045 1.0063 1.0080
Discrete Rayleigh	0.9996 0.9993 0.9998 0.9986 0.9983	0.9965 0.9932 0.9898 0.9867 0.9836	0.9940 0.9901 0.9863 0.9807 0.9761		0.9745 0.9699 0.9663 0.9612 0.9572	0.9960 0.9929 0.9907 0.9869 0.9839
Discrete Pareto	1.0257 1.0303 1.0347 1.0389 1.0429	1.0225 1.0239 1.0243 1.0265 1.0275	1.0200 1.0208 1.0207 1.0203 1.0197	1.0261 1.0310 1.0349 1.0404 1.0447		1.0220 1.0236 1.0253 1.0267 1.0278
Discrete Burr	1.0036 1.0065 1.0092 1.0119 1.0146	1.0005 1.0003 0.9991 0.9998 0.9997	0.9980 0.9973 0.9955 0.9938 0.9997	1.0040 1.0072 1.0094 1.0133 1.0164	0.9785 0.9769 0.9754 0.9740 0.9729	

Table 6 gives the level optimum premiums for annuity assuming future optimum premium to be paid by the susceptible individual or future optimum claim to be made by the infected individual follow some discrete analogues of continuous loss distributions under exponential utility function. If we assume discrete exponential for future optimum premium, the level of optimum less than unity for all distributions except discrete Rayleigh distribution and in all the cases the values are decreasing with increase in rate of interest. If we assume Gamma for future optimum premium, then the level of optimum greater than unity for discrete exponential and Rayleigh distributions, less than unity in remaining distributions and in some of the cases the values are decreasing with increase in rate of interest and in some other cases reverse.If we assume discrete Weibullor Rayleigh for future optimum premium, the level of optimum less than unity for all distributions and in all the cases the values are decreasing with increase in rate of interest.If we assume discrete Pareto or Burr for future optimum premium, the level of optimum is greater than unity for all distributions (except few values) and in all the cases the values are increasing with increase in rate of interest.

Table 7: Level discounted future premiums for the annuity using quadratic utility function for some continuous distributions.

$\bar{a}_0^i / \bar{a}_0^s$	Exponential	Gamma	Weibull	Beta	Rayleigh
Exponential		1.9989 2.0129 2.0295 2.0530 2.0834	2.1239 2.2656 2.4134 2.5754 2.7753	2.0342 2.1014 2.1792 2.2724 2.3726	2.1088 2.2362 2.3630 2.5008 2.6523
Gamma	0.5003 0.4968 0.4927 0.4871 0.4800		1.0625 1.1255 1.1892 1.2544 1.3321	1.0176 1.0440 1.0738 1.1068 1.1388	1.0549 1.1109 1.1643 1.2181 1.2730
Weibull	0.4708 0.4414 0.4143 0.3883 0.3603	0.9412 0.8885 0.8409 0.7972 0.7507		0.9578 0.9276 0.9030 0.8823 0.8549	0.9929 0.9870 0.9791 0.9710 0.9557
Beta	0.4916 0.4759 0.4589 0.4401 0.4215	0.9827 0.9579 0.9313 0.9035 0.8781	1.0441 1.0781 1.1075 1.1334 1.1697		1.0367 1.0641 1.0843 1.1005 1.1179
Rayleigh	0.4742 0.4472 0.4232 0.3999 0.3770	0.9479 0.9001 0.8589 0.8209 0.7855	1.0072 1.0131 1.0213 1.0298 1.0464	0.9646 0.9397 0.9222 0.9086 0.8946	

Table 7 gives the level optimum premiums for annuity assuming future optimum premium to be paid by the susceptible individual or future optimum claim to be made by the infected individual follow some arbitrary continuous loss distributions under quadratic utility function. If we assume exponential for future optimum premium and all other distributions for future optimum claim, the level of optimum premiums are very high. If we assume Gamma for future optimum premium, then the level of optimum slightly greater than unity for all distributions except exponential distribution and in all the cases the values are increasing with increase in rate of interest. If we assume Weibull for future optimum premium, the level of optimum slightly lower than the unity for all distributions and in all the cases the values are decreasing with increase in rate of interest. If we assume Beta for future optimum premium, the level of optimums is very less for exponential and slightly less than unity for Gamma distribution and greater than unity for Weibull and Rayleigh distributions. If we assume Rayleigh for future optimum premium, the level of optimum less than unity for all distributions except Weibull and in all the cases the values are increasing with increase in rate of interest.

Table 8: Level discounted future premiums for the annuity using quadratic utility function for some discrete analogues of continuous distributions.

$\bar{a}_0^i / \bar{a}_0^s$	Discrete Exponential	Discrete Weibull	Discrete Rayleigh	Discrete Pareto	Discrete Burr
Discrete Exponential		1.5512 1.5388 1.5130 1.5139 1.5059	1.0017 1.0035 1.0049 1.0067 1.0082	0.9599 0.9804 0.9178 0.9051 1.3364	1.0478 1.0284 1.0292 1.0139 1.0073
Discrete Weibull	0.6446 0.6498 0.6609 0.6606 0.6641		0.6458 0.6521 0.6641 0.6650 0.6695	0.6188 0.6371 0.6066 0.5979 0.8874	0.6755 0.6683 0.6802 0.6697 0.6689
Discrete Rayleigh	0.9983 0.9965 0.9952 0.9933 0.9919	1.5486 1.5335 1.5057 1.5038 1.4937		0.9583 0.9770 0.9134 0.8991 1.3255	1.0460 1.0248 1.0242 1.0071 0.9991
Discrete Pareto	1.0417 1.0200 1.0896 1.1048 0.7483	1.6160 1.5696 1.6485 1.6726 1.1269	1.0435 1.0236 1.0948 1.1123 0.7544		1.0915 1.0490 1.1213 1.1202 0.7537
Discrete Burr	0.9544 0.9724 0.9717 0.9863 0.9928	1.4805 1.4964 1.4702 1.4932 1.4950	0.9560 0.9758 0.9764 0.9929 1.0009	0.9162 0.9533 0.8918 0.8927 1.3267	

Table 8 gives the level optimum premiums for annuity assuming future optimum premium to be paid by the susceptible individual or future optimum claim to be made by the infected individual follow some arbitrary continuous loss distributions under quadratic utility function. Similar observations are made in this table. If we assume a discrete exponential for calculation of future optimum premiums, the level of optimum premium is greater than unity for all distributions except discrete Pareto distribution. If we assume discrete Pareto for future optimum premiums, the level of optimum is greater than unity for all distributions and very high in case of discrete Weibull.

4. Conclusion

If the time at which susceptible individual has to pay his future optimum premium under exponential utility function assuming loss as some continuous distributions, we may conclude that if the susceptible individual assumes $standardExponential$or$Gamma(2,1)$distribution, then the susceptible has to pay high premiums compared to different distributions and these are favorable to insurer and remaining distributions favorable to insured individuals. If the time at which susceptible individual has to pay his future optimum premium under exponential utility function assuming loss as some arbitrary discrete analogues of continuous distributions, we may conclude that if the susceptible individual assumes $discreteRayleighwith\sigma^2 = 0.25$ distribution, then the susceptible individual has to pay comparatively high premiums and this is favorable to insurer and remaining distributions favorable to insured individuals. If the time at which susceptible individual has to pay his future optimum premium under quadratic utility function assuming loss as some arbitrary continuous distributions, we may conclude that if the susceptible individual assumes$Exponential(0.5)$ has low optimal premiums and favorable to insured peoples and the remaining $Gamma(2{,}0.5), Weibull(3), Beta(2{,}1.5) and Rayleigh(1)$have favorable to insurer company. If the time at which susceptible individual has to pay his future optimum premiums under quadratic utility function assuming loss as some arbitrary discrete analogues of continuous distributions, we may conclude that, if the susceptible individual assume$discerteWeibull(2)$, individual has to pay high optimal premiums and except this remaining are beneficial to insured people and these values are very low compared to above all cases.

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