

Avariational Formulation and Navier's Method for Solving Laminated Plate Problems

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Abstract: In case of laminated plates determination of stresses and displacements under thermal loads is important as it results in delamination under thermal loads. One has to develop the equations of motion or governing equations before computing thermal deformations. A variational formulation or virtual work method is useful tool to develop the governing equations. Once the governing equations are established, Navier's method is required to apply for the determination of displacements and stresses under thermal load. A Navier's method assumes trigonometric solution that satisfy the boundary conditions. This combination (variational formulation and Navier's method) is a powerful analytical tool for solving laminated plate problems. A variational formulation is widely used in structural mechanics especially for composite laminates. Some of the structural theories like classical, first order, higher order and layer-wise theories are using virtual work method or variational formulation for laminated plates. This method consists of mainly three vital point as given below. An application of virtual work principle or virtual formulation to composite laminated plates. Use of integration by parts for further solution and development of governing equations. Apply Navier's method to obtain thermal deformations (stresses and displacements) in the laminated composite plate.

Keywords: Virtual work method, Navier's method, analytical tool, structural theories.

Background:

In the literature, there are many structural theories for the analysis of composite laminated plates under temperature field. The various methods are used to analyse the laminated plates subjected to thermal loadings. The fundamental theory known as Kirchhoff's [1] thin plate theory or classical plate theory, Mindlin's first order theory [2] and Reddy's higher order theory [3] uses virtual work method and presented in the book written by Reddy [4]. An exact method is presented by Bhaskar et.al [5] to evaluate the thermal deformations in laminated plates. Higher order theory using 12 variables has been presented by Khare et al [6] for thermal bending analysis of

laminated plates. A plate theory known as discrete layer-theory was presented by He [7] for the square plate subjected to thermal loading. The exact method was used by Reddy and Hsu [8] and transverse displacements were obtained for two-layer laminated beam. A thermal response of laminated plates was presented by Zenkour [9] by using unified shear deformation plate theory. Ali et al. [10] presented a displacement field based on higher order theory for the thermal/mechanical analysis of laminated plates.

The laminated composite plate consists of multiple layers with different orientation and properties. The virtual work method helps to derive governing equations for flexural analysis.

Method details:

The method is applied on a composite laminated plate which has the length a along x axis and width b along y axis. The thickness of the plate along z axis is h as shown in figure 1. The z axis considered as positive in downward direction.

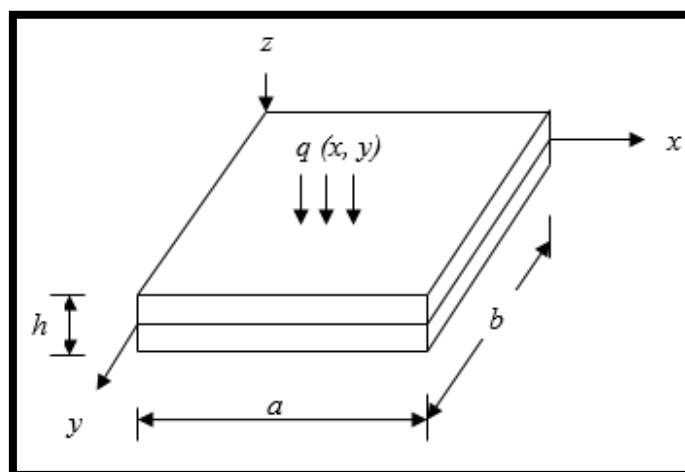


Figure 1. Plate geometry

A four variable trigonometric structural theory (FVTST) is selected and virtual work method has been applied to composite laminated plate. The displacement field of this theory has four variables and the field is represented by following equations.

$$[u(x, y, z, t)] = \left[u_0(x, y, z) - z \frac{\partial w_b(x, y)}{\partial x} - \left(z - \frac{h}{\pi} \sin \frac{\pi z}{h} \right) \frac{\partial w_s(x, y)}{\partial x} \right]$$

(1)

$$[v(x, y, z, t)] = \left[v_0(x, y, z) - z \frac{\partial w_b(x, y)}{\partial y} - \left(z - \frac{h}{\pi} \sin \frac{\pi z}{h} \right) \frac{\partial w_s(x, y)}{\partial y} \right]$$

(2)

$$[w(x, y, z, t)] = [w_b(x, y) + w_s(x, y)]$$

(3)

In the above equations u , v and w are displacements along x , y and z directions respectively. The mid-plane displacements in x and y directions are represented by u_0 and v_0 respectively. The bending and shear components of deflection are w_b and w_s respectively.

The normal strains and shear strains are computed by using following equations.

$$\left[\varepsilon_x = \frac{\partial u}{\partial x} \right], \left[\varepsilon_y = \frac{\partial v}{\partial y} \right], \left[\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right], \left[\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right], \left[\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right] \quad (4)$$

The normal stress and transverse shear stress are computed by using following stress – strain relations.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}^k = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^k \begin{Bmatrix} \varepsilon_x - \alpha_x \Delta T \\ \varepsilon_y - \alpha_y \Delta T \\ \gamma_{xy} \end{Bmatrix}^k \text{ and } \begin{Bmatrix} \tau_{xz} \\ \tau_{yz} \end{Bmatrix}^k = \begin{bmatrix} Q_{55} & 0 \\ 0 & Q_{44} \end{bmatrix}^k \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}^k \quad (5)$$

The stiffness coefficients Q_{ij} are expresses as given below.

$$\left\{ Q_{11} = \frac{E_1}{1 - \mu_{12}\mu_{21}} \right\}, \left\{ Q_{12} = \frac{\mu_{12}E_2}{1 - \mu_{12}\mu_{21}} \right\}, \left\{ Q_{22} = \frac{E_2}{1 - \mu_{12}\mu_{21}} \right\} \quad (6)$$

$$\{Q_{66} = G_{12}\}, \{Q_{55} = G_{13}\}, \{Q_{44} = G_{23}\} \quad (7)$$

E_i, G_{ij} and μ_{ij} represents the Young's modulus, shear modulus and Poisson's ratios respectively. The distribution of temperature across the thickness of composite laminated plate is considered as below.

$$\Delta T(x, y, z) = T_0(x, y) + \frac{z}{h} T_1(x, y) \quad (8)$$

In the above equation the base temperature is represented by $T_0(x, y)$ and temperature gradient across the thickness is represented by $T_1(x, y)$. The z coordinate is through the thickness from top ($-h/2$) to bottom ($+h/2$). Where h is the total thickness of the plate. This implies that the temperature varies linearly with thickness coordinate (z) and the slope of this variation is computed by $T_1(x, y)/h$. On completion of the above steps this method moves towards the use of variational principle.

This method further uses variational principles like minimum potential energy, principle of virtual work or Hamilton's principle to determine equations of motion and boundary conditions. The principle of virtual work when applied to laminated composite beams then it leads to:

$$\int_0^a \int_0^b \int_{-h/2}^{h/2} [\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} + \tau_{xz} \delta \gamma_{xz} + \tau_{yz} \delta \gamma_{yz}] dx dy dz - \int_0^a \int_0^b q(x, y) \delta w dx dy = 0 \quad (9)$$

The above equation (9) is solved by using integration by parts and final governing equations are developed as given below. In the above equation δ represents the virtual

variation. Integrating the above Eq. (9) by parts and then after collecting the coefficients of $\delta u_0, \delta v_0, \delta w_b$ and δw_s equal to zero one can evaluate the equations as below.

$$\{\delta u_0\}: -(A_{11}) \frac{\partial^2 u_0}{\partial x^2} - (A_{66}) \frac{\partial^2 u_0}{\partial y^2} - (A_{12} + A_{66}) \frac{\partial^2 v_0}{\partial x \partial y} + B_{11} \frac{\partial^3 w_b}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3 w_b}{\partial x \partial y^2} + (B_{11} - E_{11}) \frac{\partial^3 w_s}{\partial x^3} + (B_{12} - E_{12} + 2B_{66} - 2E_{66}) \frac{\partial^3 w_s}{\partial x \partial y^2} = -(B_{11}^{Tx} + B_{12}^{Ty}) \frac{1}{h} \frac{\partial T_1}{\partial x} \quad (10)$$

$$\{\delta v_0\}: -(A_{12} + A_{66}) \frac{\partial^2 u_0}{\partial x \partial y} - A_{22} \frac{\partial^2 v_0}{\partial y^2} - A_{66} \frac{\partial^2 v_0}{\partial x^2} + (B_{12} + 2B_{66}) \frac{\partial^3 w_b}{\partial x^2 \partial y} + B_{22} \frac{\partial^3 w_b}{\partial y^3} + (B_{22} - E_{22}) \frac{\partial^3 w_s}{\partial y^3} + (B_{12} - E_{12} + 2B_{66} - 2E_{66}) \frac{\partial^3 w_s}{\partial x^2 \partial y} = -(B_{12}^{Tx} + B_{22}^{Ty}) \frac{1}{h} \frac{\partial T_1}{\partial y} \quad (11)$$

$$\{\delta w_b\}: -B_{11} \frac{\partial^3 u_0}{\partial x^3} - (B_{12} + 2B_{66}) \frac{\partial^3 u_0}{\partial x \partial y^2} - (B_{12} + 2B_{66}) \frac{\partial^3 v_0}{\partial x^2 \partial y} - B_{22} \frac{\partial^3 v_0}{\partial y^3} + D_{11} \frac{\partial^4 w_b}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w_b}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w_b}{\partial y^4} + (D_{11} - S_{11}) \frac{\partial^4 w_s}{\partial x^4} + (2D_{12} - 2S_{12} + 4D_{66} - 4S_{66}) \frac{\partial^4 w_s}{\partial x^2 \partial y^2} + (D_{22} - S_{22}) \frac{\partial^4 w_s}{\partial y^4} = -(D_{11}^{Tx} + D_{12}^{Ty}) \frac{1}{h} \frac{\partial^2 T_1}{\partial x^2} - (D_{12}^{Tx} + D_{22}^{Ty}) \frac{1}{h} \frac{\partial^2 T_1}{\partial y^2} + q \quad (12)$$

$$\{\delta w_s\}: -(B_{11} - E_{11}) \frac{\partial^3 u_0}{\partial x^3} + (D_{11} - S_{11}) \frac{\partial^4 w_b}{\partial x^4} + (D_{11} + H_{11} - 2S_{11}) \frac{\partial^4 w_s}{\partial x^4} (-B_{12} + E_{12} - 2B_{66} + 2E_{66}) \frac{\partial^3 v_0}{\partial y \partial x^2} + [2(D_{12} - S_{12}) + 4(D_{66} - S_{66})] \frac{\partial^4 w_b}{\partial y^2 \partial x^2} (-B_{12} + E_{12} - 2B_{66} + 2E_{66}) \frac{\partial^3 v_0}{\partial y \partial x^2} + [2(D_{12} - S_{12}) + 4(D_{66} - S_{66})] \frac{\partial^4 w_b}{\partial y^2 \partial x^2} + [2(D_{12} + H_{12} - 2S_{12}) + 4(D_{66} + H_{66} - 2S_{66})] \frac{\partial^4 w_s}{\partial y^2 \partial x^2} + (-B_{12} + E_{12} - 2B_{66} + E_{66}) \frac{\partial^3 u_0}{\partial x \partial y^2} - (B_{22} - E_{22}) \frac{\partial^3 v_0}{\partial y^3} + (D_{22} - S_{22}) \frac{\partial^4 w_b}{\partial y^4} + (D_{22} + H_{22} - 2S_{22}) \frac{\partial^4 w_s}{\partial y^4} - C_{55} \frac{\partial^2 w_s}{\partial x^2} - C_{44} \frac{\partial^2 w_s}{\partial y^2} = (D_{11}^{Tx} - S_{11}^{Tx}) \frac{1}{h} \frac{\partial^2 T_1}{\partial x^2} - (D_{12}^{Ty} - S_{12}^{Ty}) \frac{1}{h} \frac{\partial^2 T_1}{\partial x^2} - (D_{12}^{Tx} - S_{12}^{Tx}) \frac{1}{h} \frac{\partial^2 T_1}{\partial y^2} - (D_{22}^{Ty} - S_{22}^{Ty}) \frac{1}{h} \frac{\partial^2 T_1}{\partial y^2} + q \quad (13)$$

where, the stiffness coefficients $(A_{ij}, B_{ij}, \dots)$ are as follows.

$$\{A_{ij}, B_{ij}, D_{ij}\} = \left[\sum_{k=1}^N \bar{Q}_{ij}^{(k)} \int_{h_k}^{h_{k+1}} (1, z, z^2) dz \right]$$

$$(E_{ij}, S_{ij}, H_{ij}) = \left[\sum_{k=1}^N \bar{Q}_{ij}^{(k)} \int_{h_k}^{h_{k+1}} \frac{h}{\pi} \sin \frac{\pi z}{h} \left(1, z, \frac{h}{\pi} \sin \frac{\pi z}{h} \right) dz \right]$$

$$(B_{ij}^{Tx}, D_{ij}^{Tx}, S_{ij}^{Tx}) = \left[\sum_{k=1}^N \bar{Q}_{ij}^{(k)} \alpha_x \int_{h_k}^{h_{k+1}} \left(z, z^2, z \frac{h}{\pi} \sin \frac{\pi z}{h} \right) dz \right]$$

$$(B_{ij}^{Ty}, D_{ij}^{Ty}, S_{ij}^{Ty}) = \left[\sum_{k=1}^N \bar{Q}_{ij}^{(k)} \alpha_y \int_{h_k}^{h_{k+1}} \left(z, z^2, z \frac{h}{\pi} \sin \frac{\pi z}{h} \right) dz \right]$$

$$(C_{44}, C_{55}) = \left[\sum_{k=1}^N \bar{Q}_{ij}^{(k)} \int_{h_k}^{h_{k+1}} \cos^2 \frac{\pi z}{h} dz \right]$$

Navier's Method

This Navier's solution technique is used further to obtain deformations. It is in the form of trigonometric series. On the edges of simply supported plate following boundary conditions are satisfied.

$$\text{at the edges } \{x = 0 \text{ and } x = a\}: v_0 = w_b = w_s = N_x = M_x^b = M_x^s = 0 \quad (14)$$

$$\text{at the edges } \{y = 0 \text{ and } y = b\}: u_0 = w_b = w_s = N_y = M_y^b = M_y^s = 0 \quad (15)$$

The thermal and mechanical loads are presented in double trigonometric series as given below.

$$\begin{Bmatrix} T_1 \\ q \end{Bmatrix} = \begin{Bmatrix} T_{1mn} \\ q_{mn} \end{Bmatrix} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (16)$$

The solution for (u_0, v_0, w_b, w_s) is of the form as given below which satisfies the boundary conditions given by equation (14).

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \end{Bmatrix} = \begin{Bmatrix} u_{0mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ v_{0mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ w_{bmn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\ w_{smn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{Bmatrix} \quad (17)$$

where, $(u_{0mn}, v_{0mn}, w_{bmn}, w_{smn})$ are arbitrary parameters. Substitution of this solution form into governing equations (10) – (13) of laminated plate one can obtain following equations.

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{12} & k_{22} & k_{23} & k_{24} \\ k_{13} & k_{23} & k_{33} & k_{34} \\ k_{14} & k_{24} & k_{34} & k_{44} \end{bmatrix} \begin{Bmatrix} u_{0mn} \\ v_{0mn} \\ w_{bmn} \\ w_{smn} \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \quad (18)$$

where, the elements of stiffness matrix $[K]$ are as given below.

$$K_{11} = \left[A_{11} \frac{m^2 \pi^2}{a^2} + A_{66} \frac{n^2 \pi^2}{b^2}, K_{12} = (A_{12} + A_{66}) \frac{mn \pi^2}{ab} \right]$$

$$K_{13} = \left[-B_{11} \frac{m^3 \pi^3}{a^3} - (B_{12} + 2B_{66}) \frac{mn^2 \pi^3}{ab^2} \right]$$

$$K_{14} = \left[-(B_{11} - E_{11}) \frac{m^3 \pi^3}{a^3} - (B_{12} - E_{12} + 2B_{66} - 2E_{66}) \frac{mn^2 \pi^3}{ab^2} \right]$$

$$K_{22} = \left[A_{22} \frac{n^2 \pi^2}{b^2} + A_{66} \frac{m^2 \pi^2}{a^2} \right]$$

$$K_{23} = \left[-(B_{12} + 2B_{66}) \frac{m^2 n \pi^3}{a^2 b} - B_{22} \frac{n^3 \pi^3}{b^3} \right]$$

$$K_{24} = \left[-(B_{22} - E_{22}) \frac{n^3 \pi^3}{b^3} - (B_{12} - E_{12} + 2B_{66} - 2E_{66}) \frac{m^2 n \pi^3}{a^2 b} \right]$$

$$K_{33} = \left[D_{11} \frac{m^4 \pi^4}{a^4} + (2D_{12} + 4D_{66}) \frac{m^2 n^2 \pi^4}{a^2 b^2} + D_{22} \frac{n^4 \pi^4}{b^4} \right]$$

$$K_{34} = \left[(D_{11} - S_{11}) \frac{m^4 \pi^4}{a^4} + (2D_{12} - 2S_{12} + 4D_{66} - 4S_{66}) \frac{m^2 n^2 \pi^4}{a^2 b^2} + (D_{22} - S_{22}) \frac{n^4 \pi^4}{b^4} \right]$$

$$K_{44} = \left[(D_{11} + H_{11} - 2S_{11}) \frac{m^4 \pi^4}{a^4} + (2D_{12} + 2H_{12} - 4S_{12} + 4D_{66} + 4H_{66} - 8S_{66}) \frac{m^2 n^2 \pi^4}{a^2 b^2} \right. \\ \left. + (D_{22} + H_{22} - 2S_{22}) \frac{n^4 \pi^4}{b^4} + C_{55} \frac{m^2 \pi^2}{a^2} + C_{44} \frac{n^2 \pi^2}{b^2} \right]$$

The force vector $\{f\}$ has following elements.

$$f_1 = \left[-(B_{11}^{Tx} + B_{12}^{Ty}) \frac{1}{h} T_{1mn} \frac{m\pi}{a} \right]$$

$$f_2 = \left[-(B_{12}^{Tx} - B_{22}^{Ty}) \frac{1}{h} T_{1mn} \frac{n\pi}{b} \right]$$

$$f_3 = \left[(D_{11}^{Tx} + D_{12}^{Ty}) \frac{1}{h} T_{1mn} \frac{m^2 \pi^2}{a^2} + (D_{12}^{Tx} + D_{22}^{Ty}) \frac{1}{h} T_{1mn} \frac{n^2 \pi^2}{b^2} + q \right]$$

$$f_4 = \left[(D_{11}^{Tx} - S_{11}^{Tx}) \frac{1}{h} T_{1mn} \frac{m^2 \pi^2}{a^2} + (D_{12}^{Ty} - S_{12}^{Ty}) \frac{1}{h} T_{1mn} \frac{m^2 \pi^2}{a^2} + (D_{12}^{Tx} - S_{12}^{Tx}) \frac{1}{h} T_{1mn} \frac{n^2 \pi^2}{b^2} + (D_{22}^{Ty} - S_{22}^{Ty}) \frac{1}{h} T_{1mn} \frac{n^2 \pi^2}{b^2} + q \right]$$

Once the parameters ($u_{0mn}, v_{0mn}, w_{bmn}, w_{smn}$) are computed thermal displacements and stresses can be obtained. The shear stresses are evaluated by using the equilibrium equations (19) and (20).

$$\left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \right] \quad (19)$$

$$\left[\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} = 0 \right] \quad (20)$$

Method Validation and discussion

To validate the results an example of bilayer composite laminated beam subjected to thermal field is considered and transverse shear stresses are obtained. Transverse shear stresses are very prominent in composite laminated beams. The continuity of shear stress at the interface is very necessary under thermal load. Thermal load causes delamination at the interface due to temperature effect, hence the continuity of shear stress is necessary. The following material properties of graphite-epoxy are considered in the analysis.

$$E_1 = 25 \times 10^6 \text{ psi}, E_2 = 10^6 \text{ psi}, G_{12} = G_{13} = 0.5 \times 10^6 \text{ psi}, G_{23} = 0.2 \times 10^6 \text{ psi}, \mu_{12} = \mu_{13} = 0.25$$

Transverse shear stresses ($\bar{\tau}_{xz}, \bar{\tau}_{yz}$) and transverse displacements ($\bar{w}$) are evaluated for length to thickness ratio 5 and 10. The aspect ratio is denoted by S in the table. The aspect ratio is defined as length to thickness ratio of plate ($S=a/h$). Transverse shear stresses and transverse displacements are computed by using following normalized forms.

$$(\bar{\tau}_{xz}, \bar{\tau}_{yz}) = \frac{(\tau_{xz}, \tau_{yz}) 10h}{(\alpha_1 T_1 E_2 a)}.$$

These stresses are evaluated at the point $\bar{\tau}_{xz} \left(0, \frac{b}{2}, 0\right), \bar{\tau}_{yz} \left(\frac{a}{2}, 0, 0\right)$.

$$\bar{w} \left(\frac{a}{2}, \frac{b}{2}, 0\right) = \frac{10 h w}{(\alpha_1 T_1 a^2)}$$

Table 1. Normalized stresses in two layer laminated square plate (0/90) subjected to thermal load. ($T_0 = 0, T_1 = 1, q = 0, \alpha_2 = 3\alpha_1$)

S	Model	$\bar{w}(z=0)$	$\bar{\tau}_{xz}(z=0)$	$\bar{\tau}_{yz}(z=0)$
5	FVTST (Present)	1.1421	-0.0875	-0.0875
	HOST12 [6]	1.1478	-0.0364	-0.0364
	He [7]	1.1557	-0.0740	-0.0740
	CPT [1]	1.1504	-0.0880	-0.0880
	Exact Method [8]	1.2224	-	-
10	FVTST (Present)	1.1483	-0.0220	-0.0220
	HOST12 [6]	1.1497	-0.0103	-0.0103
	He [7]	1.1519	-0.0210	-0.0210
	CPT [1]	1.1504	-0.0220	-0.0220
	Exact Method [8]	1.1365	-	-

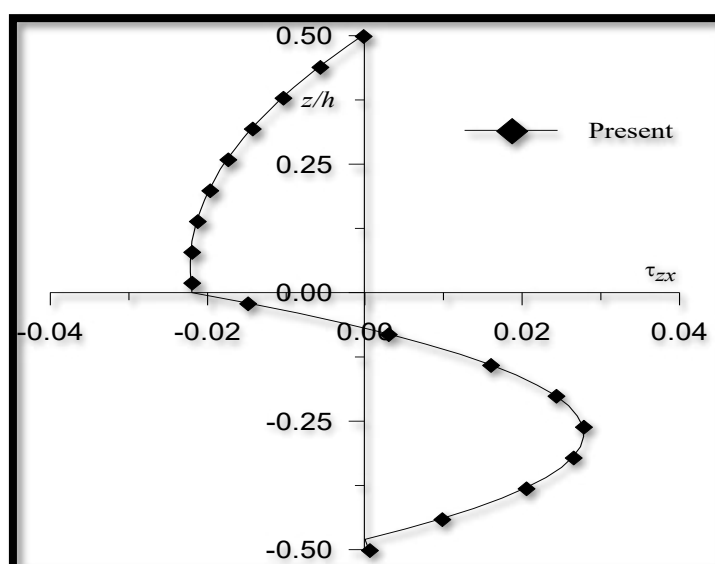


Figure 2. The continuity of shear stress at the interface

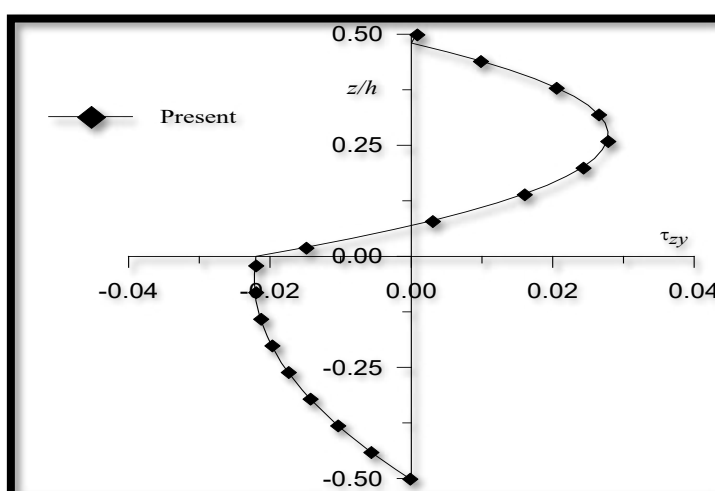


Figure 3. The continuity of shear stress at the interface

The table 1 shows the results of thermal deformations obtained by present four variable trigonometric structural theory (FVTST) by using virtual work and Navier's method. Transverse displacement ($\bar{w}$) and transverse shear stresses ($\bar{\tau}_{xz}, \bar{\tau}_{yz}$) evaluated are compared with other higher order theories. The transverse displacement ($\bar{w}$) evaluated is compared with exact method for aspect ratio 5 and 10. The percentage difference between transverse displacement obtained for aspect ratio 5 is 6.7%. When the aspect ratio changes from 5 to 10, this percentage difference reduces to 1.0%. The transverse shear stresses are continuous at the interface as shown in figures 2 and 3. This important from the point of view of delamination under thermal loading.

Conclusion

The laminated composite plates have heterogeneous properties. This method of virtual work simplifies the analysis using energy principles and by avoiding direct force calculations. This method of virtual work is versatile and applicable to various loading conditions.

References

1. G. R. Kirchhoff, Uber das Gleichgewicht and die Bewegung einer Elastischen Scheibe, *J. Reine Angew. Math. (Crelle)*, vol. 40, pp. 51-88, 1850.
2. R. D. Mindlin, Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic, Elastic Plates, *ASME J. Appl. Mech.*, vol.18, pp. 31-38, 1951.
3. J. N. Reddy, A Simple Higher Order Theory for Laminated Composite Plates, *ASME J. Appl. Mech.*, vol.51, pp. 745-752, 1984.
4. J. N. Reddy, *Mechanics of Laminated Composite Plates*, CRC Press, Boca Raton, 1997.
5. K. Bhaskar, T. K. Vardan, and J. S. M. Ali, Thermoelastic Solutions for Orthotropic and Anisotropic Composite Laminates, *Compos., B*, vol.27, pp.415-420, 1996.
6. R. K. Khare, T. Kant, and A. K. Garg, Closed-form Thermo-mechanical Solutions of Higher Order Theories of Cross-Ply Laminated Shallow Shells, *Compos. Struct.*, vol. 59, pp. 313-340, 2003
7. J. He, Thermoelastic Analysis of Laminated Plates including Transverse Shear Deformation Effects, *Compos. Struct.*, vol. 30, pp. 51-59, 1995.
8. Exact: Reddy J N, Hsu Y S. Effect of shear deformation and anisotropy on the thermal bending of layered composite plates. *Journal of Thermal Stresses* (1980), 3, 475-93.
9. A. M. Zenkour, Analytical Solution for Bending of Cross-Ply Laminated Plates Under Thermo-mechanical loading, *Compos. Struct.*, vol. 65, pp. 367-379, 2004.
10. J. S. M. Ali, K. Bhaskar, and T. K. Vardan, A new for Accurate Thermal /Mechanical Flexural Analysis of Symmetric Laminated plates, *Compos. Struct.*, vol. 45, pp. 227-232, 1999.