Adomian Polynomials & Double Elzaki Transform for Solving Some Applications of Quantum Physics

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Abstract: In this research work, we apply the computational technique based on double Elzaki transform and Adomian polynomials to predict the behavior of the solutions of fractional Schrodinger equations arising in quantum physics. The Schrodinger equation helps to describe the complex behavior of particles and energy transport in advanced materials. With the help of this hybrid technique, we obtain simple and accurate solutions without complex computations. The proposed approach can be used to study quantum properties and wave function in various materials and nanostructures, making it useful for material simulation and design.
Keywords: Double Elzaki Transform; Adomian Decomposition Method; Fractional Schrodinger Equations; Numerical Experiments.

Introduction

Linear and nonlinear PDEs play a crucial role in modeling complex physical phenomena, especially in quantum mechanics. Significant example of such equation is a fractional Schrodinger equation, which arise in the study of wave function in quantum physics. New transform known as Elzaki transform has been introduced in [1] for solving differential equations. Analytical solution of ordinary differential equations with variable coefficients has been formed with the applications of Elzaki transform in [2]. The relationship between Laplace transform and the Elzaki transform has been presented in [3] for solving differential equations. Combination of Double Elzaki transform and the decomposition method has been utilized for solving nonlinear partial differential equations in [4]. The convergence of double Elzaki transform has been presented for solving differential equations in [5]. Elzaki transform has been presented for solving Telegraph equations in [6]. Elzaki transform based Homotopy analysis method has been used for solving fractional (2+1)-D and (3+1)-D nonlinear Schrodinger equations in [7]. Two semi-analytical techniques based on integral transforms have been utilized for solving (2+1)-D and (3+1)-D Schrodinger equations in [8]. Double Elzaki transform and Adomian polynomials have been utilized for solving some models of partial differential equations in [9]. Three- dimensional Schrodinger equations have been solved with the help of new Laplace variational iteration method in [10]. Combination of Elzaki transform and the Homotopy perturbation method has been used for solving linear and nonlinear Schrodinger equations in [11]. Variational method has been used for solving linear and nonlinear Schrodinger equation in [12]. Shehu transform based Adomian decomposition method has been utilised for solving various differential equations in [13]. Double Elzaki transform based decomposition method for solving liquid drop formation models in [14]. For solving nonlinear partial differential equations, double Elzaki transform based decomposition method has been developed in [15]. Fundamental properties of Elzaki transform has been used for solving ordinary differential equations in [16].

This research paper is constituted as follow: Section 2 contains the full information regarding the double Elzaki transform and their properties. In Section 3, some computational work has been carried out by using the proposed technique for solving such equations. Section 4 contains the conclusion part of the research paper.

Double Elzaki Transform and Its Properties

Let $f(x,t)$ with $x\,,t\; > 0$ be a function. This function can be expressed in the form of an infinite series. Therefore, the double Elzaki transform is written as:

$$E_2\{f(x,t);u,v\} = T(u,v) = uv\int_0^{\infty}\int_0^{\infty} f(x,t)\, e\, -^{\left(\frac{x}{u}+\frac{t}{v}\right)}\, dxdt,$$

Whenever integral exist. The inverse of double Elzaki transform can be written as:

$$E_2{}^{-1}\{T(u,v)\} = f(x,t), \qquad x,t > 0$$

The order of the function $f(x,t)$ is said to be of exponential, if for $a > 0, b > 0$ in the region belong to the interval$0 \le x < \infty, 0 \le t < \infty$, ∃a positive constant k such that

$$|f(x,t)| \le k\, e^{\left(\frac{x}{a}+\frac{t}{b}\right)}$$

Computational Work

In this section, we will discuss about the computational framework to solve fractional Schrodinger equations. For this purpose, hybrid technique based on double Elzaki transform and the decomposition method is utilized.

Example 1: Consider the following fractional Schrodinger equation

$$-i\frac{\partial^\alpha \psi}{\partial t^\alpha} = \frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi, \qquad 0 \le \alpha \le 1 \tag{1}$$

With initial condition$\psi(x,0) = \sin \pi x$. The exact solution is $(\alpha = 1)$:

$$\psi(x,t) = e^{it} \sin \pi x$$

Equation (1) can be written as:

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} = i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}, \tag{2}$$

Taking the double Elzaki transform both sides of Equation (2), we obtain

$$E_2\left\{\frac{\partial^\alpha \psi}{\partial t^\alpha}\right\} = E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}\right\}$$

$$\frac{1}{v^\alpha}T(u,v) - \frac{1}{v^{\alpha-2}}T(u,0) = E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}\right\}$$

Using initial conditions and after simplifications, we obtain

$$T(u,v) = v^2 T(u,0) + v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}\right\} \tag{3}$$

Taking single Elzaki transform to the initial condition

$$T(u,0) = E\big(\psi(x,0)\big) = E(\sin \pi x) = \frac{\pi u^3}{1+\pi^2 u^2}$$

From (3), we obtain

$$T(u,v) = v^2 \frac{\pi u^3}{1+\pi^2 u^2} + v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}\right\}$$

Taking inverse Elzaki transform method, we obtain

$$\psi(x,t) = E_2^{-1}\left\{v^2 \frac{\pi u^3}{1+\pi^2 u^2}\right\} + E_2^{-1}\left\{v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}\right\}\right\}$$

This implies

$$\psi(x,t) = \sin \pi x + E_2^{-1}\left\{v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + (\pi^2+1)\psi\right\}\right\}\right\}$$

Applying Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin \pi x + iE_2^{-1}\left\{v^\alpha E_2\left\{\sum_{n=0}^{\infty} A_n(\psi)\right\}\right\}$$

From here, we obtain

$$\psi_0(x,t) = \sin \pi x,$$
$$\psi_1(x,t) = iE_2^{-1}\{v^\alpha E_2(A_0)\},$$
$$\psi_2(x,t) = iE_2^{-1}\{v^\alpha E_2(A_1)\},$$
$$\vdots$$

Some components of Adomian polynomials are:

$$A_0 = \frac{\partial^2 \psi_0}{\partial x^2} + (\pi^2+1)\psi_0 = \sin \pi x,$$

$$A_1 = \frac{\partial^2 \psi_1}{\partial x^2} + (\pi^2 + 1)\psi_1 = (it)\sin \pi x,$$
$$\vdots$$

Therefore, components of wave function are:

$$\begin{cases} \psi_0(x,t) = \sin \pi x, \\ \psi_1(x,t) = (it)\sin \pi x, \\ \psi_2(x,t) = \frac{(it)^2}{2!}\sin \pi x, \\ \psi_3(x,t) = \frac{(it)^3}{3!}\sin \pi x, \\ \vdots \end{cases}$$

The complete solution is:

$$\psi(x,t) = \psi_0(x,t) + \psi_1(x,t) + \psi_2(x,t) + \psi_3(x,t) + \cdots$$

This implies

$$\psi(x,t) = \left(1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right)\sin \pi x = e^{it}\sin \pi x$$

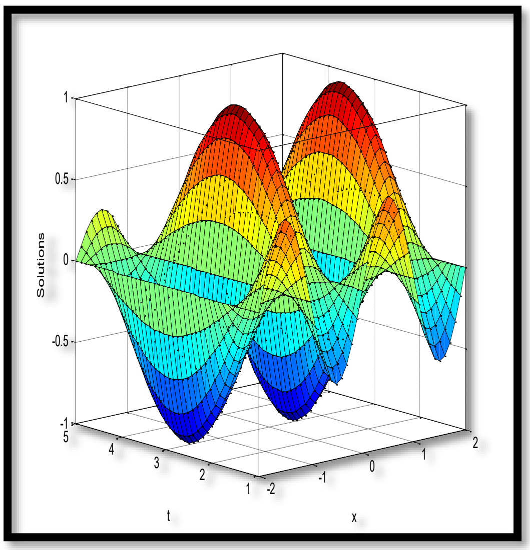

Figure (a)

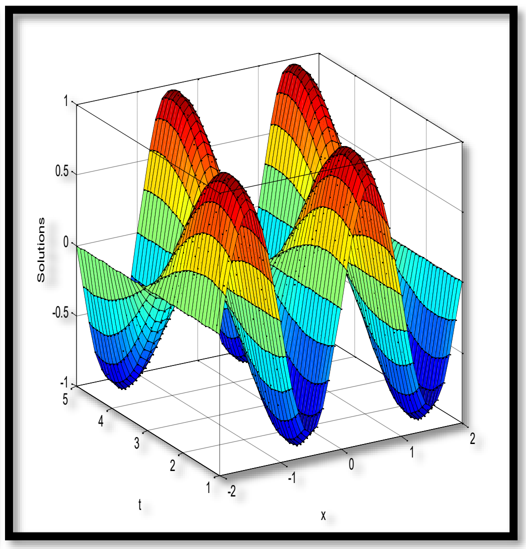

Figure (b)

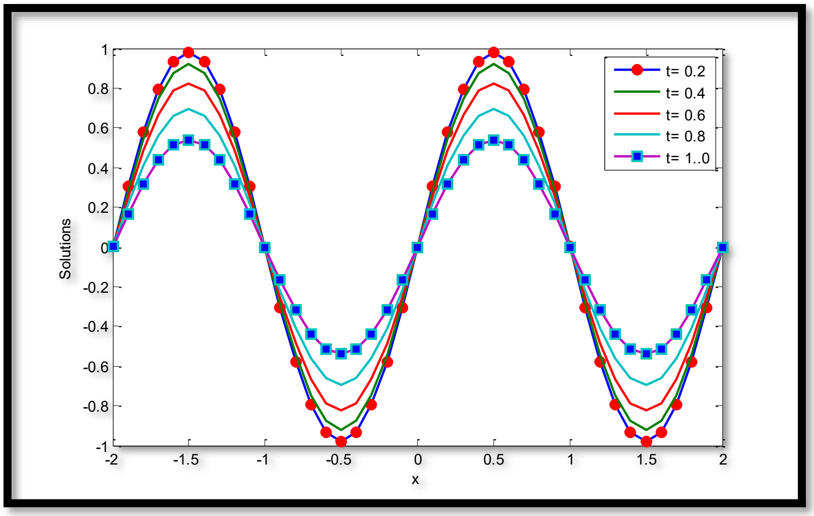

Figure (c)

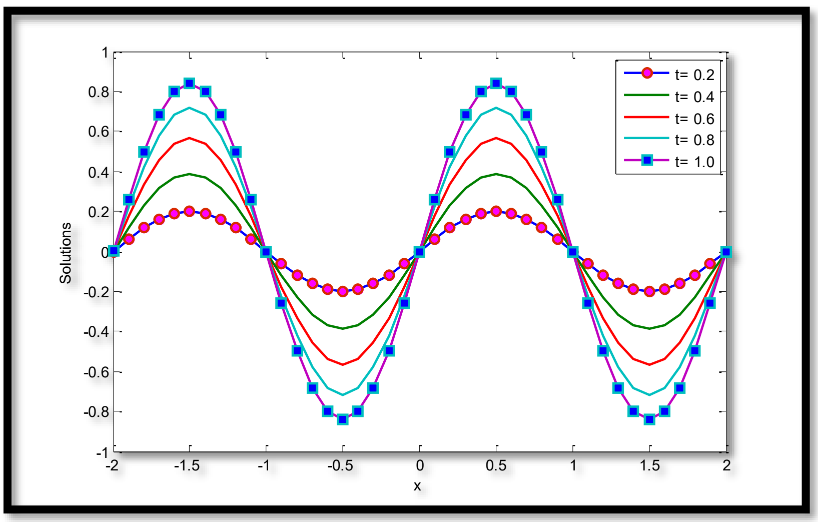

Figure (d)

Figures (a, c) show the physical behavior of real part solutions of Example 1 for different range of x and t respectively. Figures (b, d) show the physical behavior of imaginary part solutions of Example 1 for different range of x and t respectively.

Example 2: Consider the following fractional Schrodinger equation

$$-i\frac{\partial^\alpha \psi}{\partial t^\alpha} = \frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi, \qquad 0 \le \alpha \le 1 \tag{4}$$

with initial condition $\psi(x,0) = x^3(1-x)$. The exact solution is ($\alpha = 1$):

$$\psi(x,t) = e^{it}x^3(1-x)$$

Equation (1) can be written as:

$$\frac{\partial^\alpha \psi}{\partial t^\alpha} = i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}, \tag{5}$$

Taking the double Elzaki transform both sides of Equation (2), we obtain

$$E_2\left\{\frac{\partial^\alpha \psi}{\partial t^\alpha}\right\} = E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}\right\}$$

$$\frac{1}{v^\alpha}T(u,v) - \frac{1}{v^{\alpha-2}}T(u,0) = E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}\right\}$$

Using initial conditions and after simplifications, we obtain

$$T(u,v) = v^2 T(u,0) + v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}\right\} \tag{6}$$

Taking single Elzaki transform to the initial condition

$$T(u,0) = E\big(\psi(x,0)\big) = E(x^3(1-x)) = 3u^5 - 24u^6$$

From (6), we obtain

$$T(u,v) = v^2(3u^5 - 24u^6) + v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}\right\}$$

Taking inverse Elzaki transform method, we obtain

$$\psi(x,t) = E_2^{-1}\{v^2(3u^5 - 24u^6)\} + E_2^{-1}\left\{v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}\right\}\right\}$$

This implies

$$\psi(x,t) = x^3(1-x) + E_2^{-1}\left\{v^\alpha E_2\left\{i\left\{\frac{\partial^2 \psi}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi\right\}\right\}\right\}$$

Applying Adomian decomposition method, we obtain

$$\sum_{n=0}^{\infty} u_n(x,t) = x^3(1-x) + iE_2^{-1}\left\{v^\alpha E_2\left\{\sum_{n=0}^{\infty} A_n(\psi)\right\}\right\}$$

From here, we obtain

$$\psi_0(x,t) = x^3(1-x),$$
$$\psi_1(x,t) = iE_2^{-1}\{v^\alpha E_2(A_0)\},$$
$$\psi_2(x,t) = iE_2^{-1}\{v^\alpha E_2(A_1)\},$$
$$\vdots$$

Some components of Adomian polynomials are:

$$A_0 = \frac{\partial^2 \psi_0}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi_0 = x,$$

$$A_1 = \frac{\partial^2 \psi_1}{\partial x^2} + \left(1 - \frac{6(1-2x)}{x^2(1-x)}\right)\psi_1 = (it)x^3(1-x),$$

$$\vdots$$

Therefore, components of wave function are:

$$\begin{cases} \psi_0(x,t) = x^3(1-x), \\ \psi_1(x,t) = (it)x^3(1-x), \\ \psi_2(x,t) = \dfrac{(it)^2}{2!}x^3(1-x), \\ \psi_3(x,t) = \dfrac{(it)^3}{3!}x^3(1-x), \\ \quad\vdots \end{cases}$$

The complete solution is:

$$\psi(x,t) = \psi_0(x,t) + \psi_1(x,t) + \psi_2(x,t) + \psi_3(x,t) + \cdots$$

This implies

$$\psi(x,t) = \left(1 + (it) + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \cdots\right)x^3(1-x) = e^{it}x^3(1-x)$$

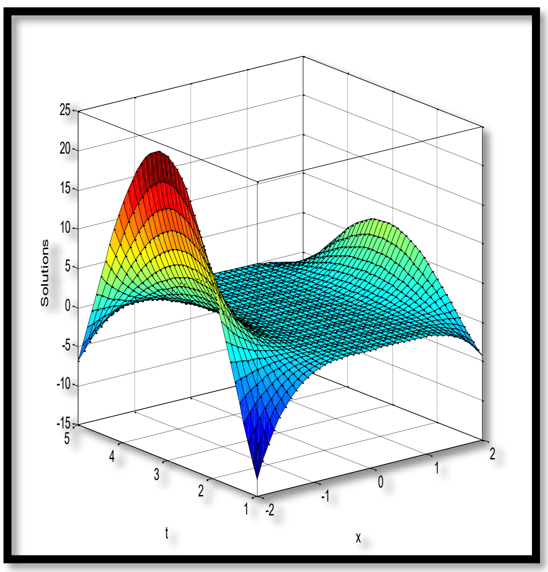

Figure (e)

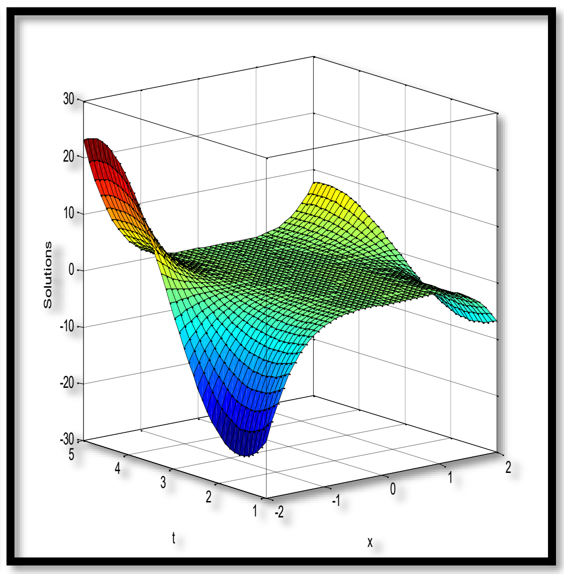

Figure (f)

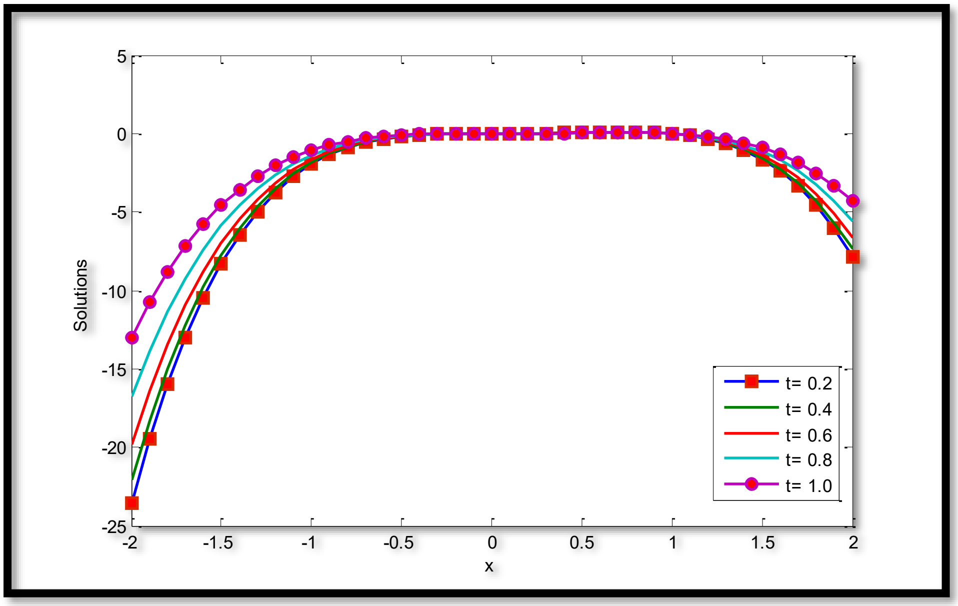

Figure (g)

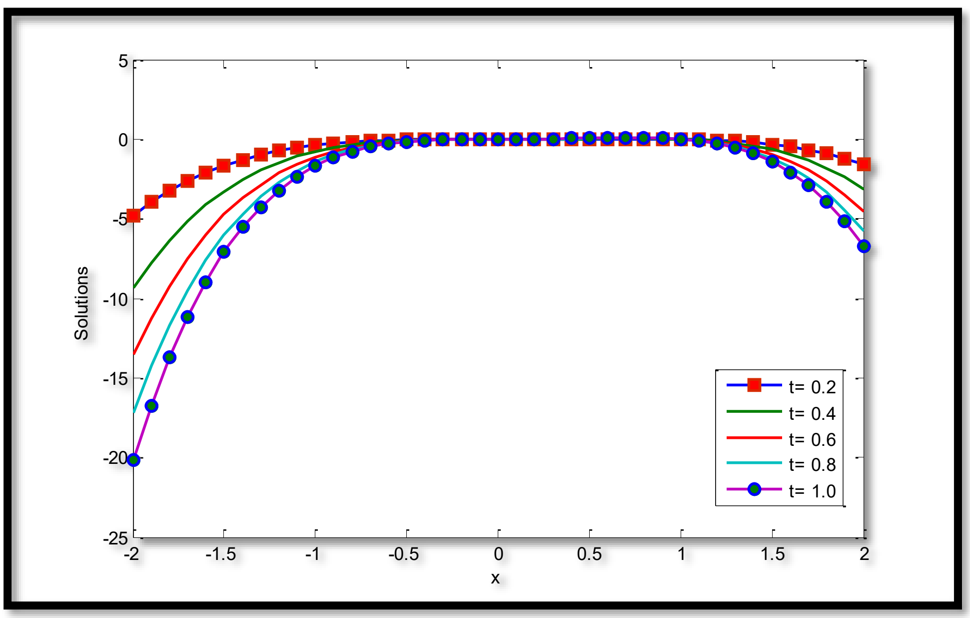

Figure (h)

Figures (e, g) show the physical behavior of real part solutions of Example 2 for different range of x and t respectively. Figures (f, h) show the physical behavior of imaginary part solutions of Example 2 for different range of x and t respectively.

5. Conclusion

From the above computational data, it is concluded that double Elzaki transform is a powerful mathematical tool, when combine with Adomian decomposition technique to predict the physical behavior of the wave function of the fractional Schrodinger equation arising in various applications of sciences and engineering. The computational results are closer to the exact results, when the terms of an infinite series may vary. For the future scope, this technique will be used to find the semi-analytical solutions of fractional nonlinear PDEs, which are arising during several applications of sciences and engineering.

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