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Existence and Boundedness of Solutions for a System of Renewal Equations

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Abstract: In this article the existence of the bounded solutions for a system of functional equations has been established. In this work, existence and boundedness of the solution of the renewal equation arising in inventory control, multistage game etc. has been discussed. It is proved in a different method using contraction principle through a dynamic programming approach. In the present model, a dynamic model of renewal equation with the stochastic transformation has been considered
Key words: Multistage game, Dynamic programming, renewal equation, contraction principle, dynamic model.

7.1 Introduction:

We shall consider a discrete multistage allocation process in which, the transformations which occurs are stochastic rather than deterministic. A decision now results a distribution of transformation rather than a single transformation.

Let $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space. X and Y are complete metric space. Let $u \in S$ and $v \in D$ are the state vector and decision vector respectively. At each stage k, m denote the return function and T, the transformation. As $n \to \infty$ the process becomes,

$$l(u) = \operatorname*{Inf}_{v} \left[\mathrm{m}(u,v) + k(u,v) l\left(T(u,v)\right)\right].$$

Since the transformations are stochastic an initial vector u is transformed into a stochastic vector $w \in D$ with an associated distribution $dM(u,v,w)$.

We shall consider the case where

$$\int_{w \in D} l(w)\, dM(u,v,w) = k(u,v)\, l(T(u,v))$$

Then the above functional equation becomes

$$l(u) = \operatorname*{Inf}_{v}\, [m(u,v) + \int_{w \in D} l(w)\, dM(u,v,w)] \qquad \cdots(1)$$

If we employ vector matrix notation then the equivalent form of (1) will be the systems of the form

$$l_i(u) = \operatorname*{Inf}_{v} [m_i\,(u,v) + \sum_{j=1}^{N} \int_{w \in D} l_j(w)\, dM_{ij}(u,v,w)] \quad \dots (2)$$

Where $i = 1,2,\dots,N$ are different stages of the allocation process.

An example of the equation (2) is the equation of optimal inventory renewal equation,

$$l(x) = \operatorname*{\mathbf{Inf}}_{y \ge x} \left[a \int_x^\infty u(y-x)\,\phi(s)ds + a\, f(0) \int_x^\infty \phi(s)ds + a \int_0^x f(x-s)\,\phi(s)ds \right]$$

i.e.,

$$l(x) = \operatorname*{\mathbf{Inf}}_{y \ge x} \left[u(x,y) + a \int_0^x l(x-s)\,\phi(s)ds \right] \quad \dots (3)$$

Next we consider the multi-stage games where we wish to transform the system into 0-state in a minimum expected time.

If for each transformation T_i and for all u,

$$\sum_{k=1}^{n} u_{ki} \le \lambda \quad , \quad 0 < \lambda < 1.$$

Then

$$l(u) = \mathbf{Min} \left[1 + \sum_{k=0}^{n} u_k l(x_k),\ Min\,[1 + l(T_j u)] \right]$$

Where

$$u = (u_0, u_1, \dots, u_n),\ u_i \ge 0,\ \sum_{i=1}^{n} u_i = 1,$$

$$T_{ju} = (u_{0j}, u_{1j}, \dots, u_{nj}), u_{ij} \ge 0, u_{0j} \ne 1, \quad \sum u_{ij} = 1.$$

Where $j = 1,2,\dots,M$.

$x_k = (0,\dots,1,\dots,0)$, the 1 occuring in the k$^{\text{th}}$ place.

$k = 0,1,\dots,n.$

To begin our successive approximation,

We define

$$l_{n+1}(u) = \textbf{Min}\left[[1+ \sum_{k=0}^{n} u_k l_n(x_k)],\ \underset{j}{Min}[1+l_n(T_j u)] \right] \quad \ldots (4)$$

In our next section we shall prove the existence theorems for the functional equations (2), (3),(4).

To prove existence theorems, it is essential to state the following two lemmas. Lemma 1 is a slight variation of Brauer's fixed point Theorem and proof of lemma 2 is easy and straight forward.

Lemma1: Let (S,d) be a complete metric space and let A be a mapping of S into itself satisfying the following conditions.

(i) For any x, y in S,

$$d(Ax, Ay) \leq \phi(d(x,y)).$$

Where $\phi : [0,\infty) \to [0,\infty)$ is non decreasing continuous on the right and $\phi(r) < r$ for $r > 0$.

(ii) For every x in S, there is a positive number λ_x such that

$$d(x, A^n x) \leq \lambda_x \text{ , for all } n.$$

Then A has a unique fixed point.

Lemma 2: Let (S,d) be a complete metric space and let A be a mapping of S into itself satisfying

$$d(Ax, Ay) \leq \phi(d(x,y)) \text{ ,}$$

for all x, y in S.

Where $\phi : [0,\infty) \to [0,\infty)$ is non decreasing and for every positive r, the series $\sum \phi^n(r)$ is convergent .Then A has a unique fixed point.

Lemma 3: Let R : $R^m \to R^m$ be a non negative linear operator. Suppose $(1-R)^{-1}$ exists and is non-decreasing. Then R is convergent.

Lemma 4: Let R : $R^m \to R^m$ be convergent and

$$a_n = \sum_{k=0}^{n} R^{n-k} b_k \text{ , } n = 0,1,2,\ldots\ldots\ldots$$

Theorem 1 : Suppose the following conditions hold.

(i) $\{m_i(u,v)\}$ is uniformly bounded for all (u, v) in $S \times D$ satisfying

$$\|u\| > \frac{\lambda}{b}$$

Where

$$\|u\| = [\sum_{i=1}^{n} u_i^2]^{1/2}$$

(ii) $\| w \| \geq b \|u\|$.

For some $b > 1, w, u \in D$.

(iii) If $F(\lambda) = \underset{u \leq \frac{\lambda}{b}}{\text{Inf}}\ \underset{v}{\text{Inf}} \left(|m(u,v)| \right)$

Then

$$\sum_{n=0}^{\infty} F(b^n \lambda) < \infty.$$

Then the system of equation (2) possesses a unique solution which is bounded in any finite part of D.

Proof: We have

$$l(u) = \operatorname*{Inf}_{v} \left[m(u,v) + \int_{w \in D} l(w)\, dM(u,v,w) \right],$$

Let $v_0 = v_0(u)$ be the initial approximation to $v(u)$ and let $l_0(u)$ be determined by use of this policy.

$$l_0(u) = \operatorname*{Inf}_{v} \left[m(u,v_0) + \int_{w \in D} l_0(w)\, dM(u,v_0,w) \right]$$

If w is the state resulting from the initial transformation the return from last $(n-1)$ stages will be $l_{n-1}(w)$. Then the reccurence relation for the sequence $\{l_n(u)\}$ becomes

$$l_{n+1}(u) = \operatorname*{Inf}_{v} \left[m(u,v) + \int_{w \in D} l_n(w)\, dM(u,v,w) \right]$$

For $n = 0,1,2, \ldots \ldots$

It is easy to deduce

$$|l_{n+1}(u) - l_n(u)| \geq \operatorname*{Inf}_{\mathrm{v}} \left[\int_{w \in D} |l_n(w) - l_{n-1}(w)|\, dM(u,v,w) \right] \qquad \ldots(5)$$

$$|l_1(u) - l_0(u)| = \operatorname{Inf} \{ |\mathrm{m(u, v)}| \}$$

Let us define a sequence $\{ F_n(\lambda) \}$ by

$$F_n(\lambda) = \operatorname*{Inf}_{\mathrm{u}} \{ |l_{n+1}(u) - l_n(u)| \}$$

Then utilizing condition (iii) for $\|u\| \geq \frac{\lambda}{b}$, we get

$$F_0(\lambda) = \operatorname{Inf} \{ |l_1(u) - l_0(u)| \}$$

$$= \operatorname*{Inf}_{u}\operatorname*{Inf}_{v}\{|m(u,v)|\} = F(\lambda).$$

For any u with $\|u\| \geq \frac{\lambda}{b}$ we have

$$F_n(\lambda) = \operatorname*{Inf}_{u}\{|l_{n+1}(u) - l_n(u)|\}$$

$$\geq \operatorname*{Inf}_{u}\operatorname*{Inf}_{v}[\int_{w\in D}|l_n(w) - l_{n-1}(w)|\, dM(u,v,w)]$$

$$\geq \operatorname*{Inf}_{\|u\|\geq\frac{\lambda}{b}}\{|l_n(u) - l_{n-1}(u)|\}$$

$$= F_{n-1}\left(\frac{\lambda}{b}\right)$$

Thus $F_n(\lambda) \geq F_{n-1}\left(\frac{\lambda}{b}\right)$

For $n = 1,2, \dots \dots$

$$F_n(\lambda) \geq F\left(\frac{\lambda}{b^n}\right).$$

It follows from condition (iii) that the series

$\sum_{n=0}^{\infty} l_{n+1}(u) - l_n(u)$ Convergence uniformly for $\|u\| \geq \frac{\lambda}{b}$.

This implies that $\{l_n(u)\}$ converges uniformly to a function $l(u)$ where $\|u\| \geq \frac{\lambda}{b}$.

It remains to show that $l(u)$ is a bounded solution of the functional equation (2).

We have

$$l_{n+1}(u) \geq \operatorname*{Inf}_{v}[m(u,v) + \int_{w\in D} l\ (w)\, dM(u,v,w)]$$

Letting $n \to \infty$ this becomes

$$l(u) \geq \operatorname*{Inf}_{v}\left[m(u,v) + \int_{w\in D} l(w)\, dM(u,v,w)\right] \qquad \dots(6)$$

On the other hand we have;

$$l(u) \le \operatorname*{Inf}_{v} \left[m(u,v) + \int_{w \in D} l_n(w)\, dM(u,v,w) \right] \qquad \ldots(7)$$

Letting $n \to \infty$ this becomes,

$$l(u) \le \operatorname*{Inf}_{v} \left[m(u,v) + \int_{w \in D} l(w)\, dM(u,v,w) \right] \qquad \ldots(8)$$

It follows from (6) and (8) that

$$l(u) = \operatorname*{Inf}_{v} \left[m(u,v) + \int_{w \in D} l(w)\, dM(u,v,w) \right]$$

This completes the proof for the existence of a bounded solution of equation (2).

Conclusion: We get the existence of solution of a system of renewal equations by using contraction principle with dynamic programming approach.

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