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The Evolution of Wavelets: A Comprehensive Historical Review

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Abstract: This article reviews the historical development of wavelets. We have presented that how wavelet transform overcomes the drawbacks of Fourier transform. The concept of Multiresolution analysis to form ortho normal wavelets and some generalizations of wavelets are also discussed.
Keywords: Fourier transform, window Fourier transform, short-time Fourier transform, Wavelet transform, Multi resolution analysis, Wave packets.

Introduction

Wavelet transform can be considered as a refinement of Fourier transform. Fourier transform is used in signal processing for transforming the time domain signal to frequency domain. It provides only frequency resolution of signal but does not provide time resolution. Fourier analysis is not useful for those signals whose frequency vary with time. For example, speech, music, images and medical signals have changing frequency. Fourier transform is also not applicable for the signals which contain discontinuities and sharp spikes. Wavelet transform overcomes the shortcomings of the Fourier transform. Wavelets can examine signals simultaneously in both time and frequency domain [1]. Wavelet transform is useful for aperiodic, transient and intermittent signals. Wavelets have several useful properties such as orthogonality, compact support, exact representation of polynomials to a certain degree and ability to represent functions at different levels of resolution. Wavelet analysis and its applications have become one of the fastest growing research areas in recent years. It has applications in the field of data compression, computer graphics, signal processing, numerical analysis, time-frequency analysis, pattern recognition, image processing, data mining and other medical image technology like EEG, ECG etc. The numerical methods based on wavelets have been developed for the solution of differential equations, integral equations, integro-differential equations, partial differential equations, fuzzy integro-differential equationsetc [2]. In these numerical methods different kind of wavelet families such asHaar [3], Chebyshev [4], Taylor [5] etc. have been used.

1. Historical Development

Wavelet analysis is a rapidly developing area of mathematics with applications in the field of science and engineering. The concept of wavelet analysis can be viewed as the synthesis of various ideas originated from different disciplines including mathematics, physics and engineering (e.g. Littlewood-Paley theory, Calderón Zygmund operators, quadrature mirror filters, sideband coding in signal processing, pyramidal algorithms in image processing, coherent states in quantum mechanics and renormalization theory).
Wavelet theory has gained present growth through the work of Strömberg (1981), Grossman and Morlet (1984), Battle(1987), Mallat (1988), Chui (1992), Coifman et al. (1992), Daubechies (1992), Meyer (1992)[1, 6-12]. Many other researchers have also made significant contributions. Wavelet transform deals with decomposition of signal or function in terms of basis functions. These basis functions are generated by applying translations and dilations operations on the mother wavelet function. Wavelets have applications in financial time series analysis, climate analysis, heart monitoring, seismic signal denoising, audio and video compression, computer graphics, compression of medical and thumb impression records, numerical analysis etc.
Wavelet analysis can be regarded as a generalization of Fourier analysis. Fourier analysis has two components: Fourier series and Fourier transform. Fourier series is the representation of a periodic signal into a series of basis functions (sine and cosine). For non-periodic functions the Fourier series is converted to Fourier integral. The complex form of Fourier integral is known as Fourier transform. In Fourier transform, signals are assumed infinite in time and in Fourier series signal are assumed periodic. But in real life we deal with signals which are of short duration and non-periodic. Also, in Fourier analysis entire Fourier spectrum is affected by a change in signal in small neighbourhood of a particular time. This means that in Fourier transform for integration we require complete description of the signal over the whole of the real line $(-\infty, \infty)$.
D. Gabor was the first who observed the drawback of Fourier transform in time-frequency analysis in 1946 [13] and used the concept of window function to define another transform. He used the Gaussian function as window function and defined the Gabor transform. This Gabor transform can be generalized to any other window Fourier transform by using any window function. This time-frequency analysis method is called short-time Fourier transform (STFT) or windowed Fourier transform. In STFT, we first select the desired portion of the given signal and then take the Fourier transform of that part. The drawback of STFT is that it uses a single window for all frequencies and therefore the resolution of analysis is same at all locations in time-frequency plane. Due to Heisenberg uncertainty principle, a window of large width provides good frequency resolution but poor time resolution and a small window width provides good time

resolution and poor frequency resolution. So, we cannot get both good time and good frequency resolution simultaneously.

Wavelet transform overcomes the drawbacks of STFT. As we know in STFT the length of the window function remains same during analysis of selected portion of signal but in wavelet transform we build a family of window functions of different length by translation and dilation of mother wavelet. Wavelets are effective in representing nonstationary (transient) signals. Unlike Fourier analysis that uses nonlocal functions as basis, wavelet analysis has basis which are localized in both time and frequency to represent nonstationary signals. Wavelets can examine signals simultaneously in time and frequency domain. Wavelet transform is useful for aperiodic, transient, noisy and intermittent signals.

In 1910, Hungarian mathematician Alfred Haar introduced a system of functions which form orthonormal basis for $L^2(R)$ and are constructed by dilations and translations of simple piecewise constant function. This system of functions is now known as Haar wavelets [14]. In 1982, Morlet defines wavelets as a family generated by applying translation and dilation operations on a singlefunction known as mother wavelet [15, 16]. For$\psi(x) \in L^2(R)$ known as mother wavelet, the family of wavelets$\psi_{a,b}(x)$ are defined as

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\psi\left(\frac{x-b}{a}\right); \;\; a,b \in R, a \neq 0 \qquad (1.1.1)$$

By reducing a, the support of $\psi_{a,b}(x)$ is reduced in time and hence covers a large frequency range and vice versa. So $1/a$is a measure of frequency. The parameter $b$indicates the location of the wavelet window along the time axis.

Grossman worked with Morlet to confirm that waves could be reconstructed from their wavelet decompositions and introduced mathematical formulation of wavelet transform and inverse wavelet transform [17].

Meyer (1985) constructed orthogonal wavelets in which the information captured by one wavelet is completely independent of the information captured by another wavelet [18]. Lemarié and Meyer (1986) constructed smooth orthonormal basis in R which have been very useful in image processing, signal processing, quantum field theory and computer vision [19]. Daubechies et al. (1986) introduced the nonorthogonal wavelet expansion [20]. Battle (1987) and Lemarié (1988, 1989) introducedspline orthogonal wavelets [21, 22]. Orthogonal wavelets have been constructed byMeyer (1989) [24] and Mallat (1989a,1989b) by using multiresolution analysis (MRA) [25, 26].

Using the MRA, Daubechies (1988) constructed the compactly supported orthonormalwavelets [27]. Wojtaszczyk (1997) extended the theory of MRA to higherdimensions by using matrix dilations [28].

Xia and Suter (1996) introduced vector-valued multiresolution analysis andvector-valued wavelets for vector-valued signal spaces [29]. Vector-valued wavelets andwavelet packets which are orthogonal and of compact support are introduced byChen and Cheng (2007) [30]. Farkov (2005) constructed compactly supported orthogonal

p −wavelets on $L^2(R_+)$ [31]. Orthogonal vector-valued wavelets on R_+ has beenconstructed by Manchanda and Sharma (2012) [32]. Manchanda and Sharma (2014)have also constructed vector-valued wavelet packets on R_+ using Walsh Fouriertransform [33]. Meenakshi et al. (2012) have introduced nonuniform multiresolutionanalysis on R_+ [34]. Meenakshi et al. (2014) have obtained wavelets from vector-valuednonuniform multiresolution analysis [35].Candés and Donoho (2000) have introduced a new extension of wavelet transform,called curvelet transform [36]. Curvelet transform is a high dimensional generalizationof wavelet transform which represents images at different scales anddifferent orientations [Candés et al. (2006)][37]. It has time-frequency localizationproperties of wavelets and also shows a high degree of directionality and anisotropy.Coifman et al. (1992) constructed the wavelet packets [38]. Wavelet packets are ageneralization of wavelets and the frequency resolution of wavelet packets are superiorthan the wavelets. They are better in representing oscillatory or periodic signals.

Wavelet packets are superposition or linear combination of wavelets and formbases which retain properties of smoothness, localisation and orthogonality fromtheir parent wavelets [39]. Wickerhauser (1994)has studied discrete wavelet packets and developed computer programmes to implementthem [40]. Cohen and Daubechies (1993) introduced biorthogonal waveletpackets [41]. Long and Chen (1997) introduced non-separable wavelet packets on R^d[42].

Chui and Li (1993) studied non-orthogonal wavelet packets [43]. Quak and Weyrich(1997a) have introduced periodic wavelet packets [44]. Quak and Weyrich (1997b)have also studied spline wavelet packets on a closed interval [45]. Lian (2004) introducedwavelet and wavelet packet associated with dilation matrices[46]. On local fieldof positive characteristic, the construction of wavelet packets and wavelet framepackets was reported by Behera and Jahan (2012) [47].

Duffin and Schaeffer (1952) introduced frames for studying nonharmonic Fourierseries [48].Frames provide us an alternative to orthonormal or Riesz basis in Hilbertspace. Daubechies et al. (1986) connects frames with wavelets and Gabor system [49].Daubechies (1992) has introduced the necessary and sufficient condition for constructionof wavelet frames [11]and then the improved result has been given by Chuiand Shi (1993) [50]. The sufficient condition for generating orthogonal wavelet frameis given by Bhatt et al. (2007) [51]. The dyadic wavelet frames on positive half-lineR_+has been constructed by Shah and Debnath (2011) by using walsh Fouriertransform [52]. Sharma and Manchanda (2015a) have introduced nonuniform waveletframes in$L_2(R)$ [53].

Wave packets, an extension of wavelets are obtained by translation, modulationand dilation of a single or finite set of functions in$L^2(R)$and were firstly used byCórdoba and Fefferman (1978) to study singular integrable operators [54]. The continuousand discrete wave packet systems in$L^2(R^d)$ have been studied by Labateet al. (2004) [55]. Wu et al. (2014) have introduced the necessary and sufficient conditionsof the wave packet frames in$L^2(R^n)$[56]. Non uniform wave packet frames in$L^2(R)$ have been studied by Sharma and Manchanda (2015b) [57].

2. Wavelet Transform

Definition (Wavelet). A function $\psi(x) \in L^2(R)$is called a wavelet if it satisfies the following properties [11]:

1. $\int_{-\infty}^{\infty} \psi(x)dx = 0,$
2. $C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(w)|^2}{|w|} dw < \infty,$

where $\hat{\psi}(w)$ is the Fourier transform of $\psi(x)$ and C_ψ being the admissibility constant. The condition (2) is known as admissibility condition.

Examples of wavelets:

Example 2.1 (**Haar wavelet**). Haar wavelet is the simplest wavelet defined on real line by [14]

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}; \\ -1, & \frac{1}{2} \le x < 1; \\ 0, & elsewhere. \end{cases}$$

The function $\psi(x)$is zero outside the interval [0,1] (Figure 1.1). So, Haar wavelet has compact support.

It has good time localisation but poor frequency localisation due to the discontinuity of $\psi(x)$.

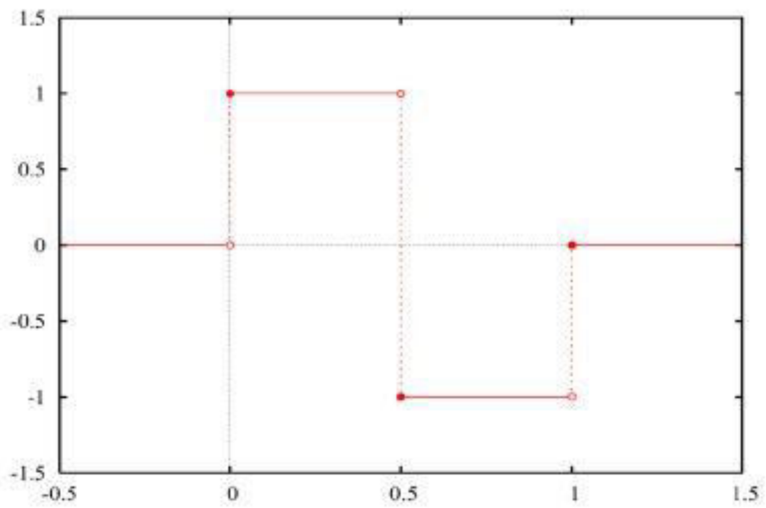

Figure 1.1: Haar wavelet

Example 2.2 (Mexican hat wavelet). Mexican hat wavelet is given by

$$\psi(x) = -\frac{d^2}{dx^2}\left(e^{-x^2/2}\right) = (1 - x^2)e^{-x^2/2}$$

It is the second derivative of the Gaussian function $e^{-x^2/2}$. Its graphical representation is given in figure 1.2. Mexican hat wavelet has good localisation in both time and frequency domains. This wavelet has two vanishing moments.

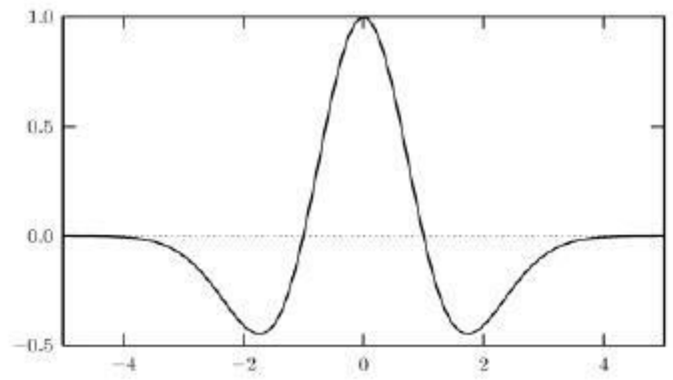

Figure 1.2: Mexican hat wavelet

Example 2.3 (Daubechies wavelet). Daubechies wavelets are a family of compactlysupported orthogonal wavelets. These wavelets have maximal number of vanishing moments as compared to other wavelets. Daubechies wavelets are represented by DN where N denotes the number of scaling coefficients. Daubechies with $N = 2$ vanishing moments has N coefficients. Daubechies wavelets with one vanishing moment i.e. D2 are the Haar wavelets. Daubechies wavelets cannot be expressed in closed form.

Definition2.1 (Continuous wavelet transform). Continuous wavelet transform of a square integrable function $g(x)$ with respect to a wavelet $\psi(x)$ is given by[7]

$$W_\psi g(a,b) = \frac{1}{\sqrt{|a|}}\int_{-\infty}^{\infty} g(x)\overline{\psi\left(\frac{x-b}{a}\right)}\,dx$$

where $a, b \in R, a \neq 0$ and $\overline{\psi(x)}$ is the complex conjugate of $\psi(x)$.

Definition2.2(Discrete wavelet transform). In Discrete wavelet transform a function is represented by a countable set of wavelet coefficients. It is an implementation of the wavelet transform using a discrete set of the wavelet scales and translation. We take $a = a_0{}^j$, $b = kb_0a_0{}^j$ and $a_0 > 1, b_0 > 0$ are fixed. So (1.1.1) becomes [1]

$$\psi_{j,k}(x) = a_0{}^{-j/2}\psi\left(a_0{}^{-j}x - kb_0\right), \quad j,k \in Z$$

For computational efficiency, we take $a_0 = 2$ and $b_0 = 1$. Therefore

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)\,, \quad j,k \in Z$$

Definition 2.3(Inverse wavelet transform). Let $\psi(x)$ be a wavelet and$W_\psi g(a,b)$ is the wavelet transform of $g(x) \in L^2(R)$with respect to$\psi(x)$. The originalfunction $g(x)$ is reconstructed from its integral wavelet transform by inversewavelet transform. The expression for inverse wavelet transform is given by

$$g(x) = \frac{1}{C_\psi}\int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} \frac{1}{a^2}\left(W_\psi g(a,b)\right)\psi_{a,b}(x)da,$$

where $C_\psi$is admissibility constant.

Definition 2.4 (Wavelet family): A function $\psi(x) \in L^2(R)$ is called orthonormal wavelet if $\psi_{j,k}(x)$ given by

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k), \quad j,k \in Z$$

form orthonormal basis in $L^2(R)$. $\{\psi_{j,k}(x)\}_{j,k\in Z}$ is called wavelet family.

Definition 2.5 (Wavelet Coefficients): Wavelet Coefficients $h_{j,k}$ of a function $g(x) \in L^2(R)$ are) given by

$$h_{j,k} =< g,\psi_{j,k} >= \int_{-\infty}^{\infty} g(x)\psi_{j,k}dx.$$

Definition 2.6 (Wavelet series): The series $\sum_{j\in Z}\sum_{k\in Z} < g,\psi_{j,k} > \psi_{j,k}$ is called wavelet series of $g(x)$. The expression given by

$$g(x) = \sum_{j\in Z}\sum_{k\in Z} < g,\psi_{j,k} > \psi_{j,k}$$

is wavelet representation of $g(x)$.

3. Multiresolution Analysis (MRA)

Multiresolution analysis is a method for constructing orthonormal wavelets. In MRA the whole function space $L^2(R)$ is decomposed into subspaces at different scales.

Definition 1.3.15. A multiresolution analysis (MRA) is a sequence of subspaces $\{V_j\}_{j\in Z}$ of $L^2(R)$with the following axioms [11]:

1. $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots,$
2. $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(R),$
3. $\bigcap_{j=-\infty}^{\infty} V_j = \{0\},$
4. $g(x) \in V_0$ if ad only if $g(x-k) \in V_0\ \forall\, k \in Z,$
5. $g(x) \in V_0$ if ad only if $D_{2^j}g(x) \in V_j\ \ \forall\, j \in Z,$
6. There exists a scaling function $\phi(x)$ in V_0 such that $\{\phi(x-k): k \in Z\}$ is an orthonormal basis for V_0.

Since $\{\phi(x-k): k \in Z\}$ is an orthonormal basis for V_0 and by property (5) of MRA, $\left\{\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)\middle| k \in Z\right\}$ is an orthonormal basis of V_j. For $j = 1$,
$\phi_{1,k}(x) = 2^{1/2}\phi(2x-k) | k \in Z$ is an orthonormal basis of V_1.
Now $\phi_{0,0}(x) = \phi(x) \subset V_0 \subset V_1$
Hence $\phi(x) = \sqrt{2}\sum_{k=-\infty}^{\infty} h_k \phi(2x-k)$
This equation is called dilation equation (two scale relation for scaling function) and the coefficients h_k are called the low pass filter coefficients.
For given nested sequence of approximation subspaces V_j we now define the detail space W_j as the orthogonal complement of V_j in V_{j+1} i.e.

$$V_j \perp W_j$$

and

$$V_{j+1} = V_j \oplus W_j \quad (1.3.1)$$

Applying (1.3.1) recursively,

$$V_j = V_{j-1} \oplus W_{j-1}$$

$$V_j = V_{j-2} \oplus W_{j-2} \oplus W_{j-1}$$

$$V_J = V_{J_0} \oplus W_{J_0} \oplus W_{J_1} \oplus W_{J_2} \oplus \dots \oplus W_{J-1}, \qquad J > J_0 \quad (1.3.2)$$

Therefore, we can analyze a function of V_j at different scales using relation (1.3.2).
On continuing the above decomposition of (1.3.2) for $J_0 \to -\infty$ and $J \to \infty$, we get

$$\oplus_{-\infty}^{\infty} W_j = L^2(R)$$

The information in moving from V_0 to V_1 is captured by translation of $\psi(x)$. So, for a given MRA there always exist a function $\psi_{0,0}(x) = \psi(x) \in W_0$ (called mother wavelet) such that

$$\left\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)\middle| k \in Z\right\}$$

is an orthonormal basis of W_j.
Now

$$\psi(x) \in W_0 \subset V_1$$

so

$$\psi(x) = \sqrt{2}\sum_{k=-\infty}^{\infty} g_k \phi(2x-k).$$

The above equation is called wavelet equation (two scale relation for wavelet function).
The coefficients g_k are called high pass filter coefficients.
So, mother wavelet function $\psi(x)$ can be constructed by MRA using sequences of subspaces V_j of $L^2(R)$ (called approximation spaces) that satisfies the properties (1)-(6) of MRA and sequence of subspaces W_j (detail spaces) of $L^2(R)$ which satisfy relation (1.3.1).

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