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On Optimization of Sub-differentiable Lower Semi-continuous Pseudo convex Functions on Banach Spaces Using Variational Inequalities

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Abstract
In this work, we extend the relationships between convex functions and corresponding monotone maps to pseudo-convexity and the corresponding pseudo-monotonicity of their sub-differentiable maps. We characterize the lower semi-continuous Clarke-Rockafeller sub-differentiable pseudo-convex functions $f: C \subseteq X \to \mathbb{R} \cup \{+\infty\}$ on convex subset C of infinite-dimensional real Banach spaces X with respect to the corresponding monotonicity of their Clarke-Rockafeller sub-differential operators ∂f and obtain the optimality conditions, using variational inequalities.
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1. Introduction

Convex optimization, which studies the problem of minimizing convex functions over convex sets, and a subfield of mathematical optimization, plays an important role in many branches of applied mathematics. The foremost reason is that; it is very suitable to extremum problems. For instance, some necessary conditions for the existence of a minimum also become sufficient in the in terms of convexity. And convex optimization can be a smooth or a non-smooth convex optimization. Since the concept of convexity does not satisfy some mathematical models, various generalizations of convexity such as quasi-convexity and pseudo-convexity, which retain some important properties of convexity and equally provide a better representation of reality were introduced in the literature to fill these gaps.

While the quasi-convexity property of a function guarantees the convexity of their sublevel sets, the pseudo-convexity property implies that the critical points are minimizers [25]. One of the features of convexity of functions is the relationship it has with the monotonicity of some maps. For example, a differentiable function is said to be convex if and only if its gradient is a monotone map. In non-smooth analysis, the generalized convexity of functions can be equally characterized in terms of the generalized monotonicity of their related operators, [12].

The concepts of pseudo-convexity, traced to [30], within his research on analytical functions and independently introduced into the field of optimization by [32], has many applications in mathematical programming and economic problems [20, 22, 29]. And pseudo-monotonicity, introduced by [24] as a generalization of monotone operators has been used to describe a property of consumer's demand correspondence [18]. Although the simplest class of pseudo-monotone operators consists of gradients of pseudo-convex functions, there are some monotone operators that are not sub-differentials, [18]. And generalized monotonicity of maps is frequently used in complementarity problems, equilibrium problems and variational inequalities [17].

Variational inequalities, formulated between late 60s and early 70s by an Italian Mathematician, Stampacchia, solve a wide range of problems in mathematical optimization, operations research, economic equilibrium problems and engineering sciences [11, 15, 37]. Results on relations of variational inequalities with differentiable optimization problems also show that the Stampacchia Variational Inequality (SVI) is a necessary condition for optimality, while the Minty Variational Inequality (MVI) is a sufficient optimality condition and similar result on generalizations of SVI and MVI to multivalued operators for non-smooth optimization problems exists [9].

Consider an optimization problem

$$minimize\ f(x), \text{ subject to } x \in C. \tag{1}$$

where $f: C \subseteq X \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous (l.s.c.) Clarke-Rockafeller sub-differentiable function on as subset C of a real Banach space X. We characterize (1) by the corresponding monotonicity of their Clarke-Rockafeller sub-differential operator ∂f and investigate their optimality conditions, using variational inequalities.

2. Preliminaries

Let X be a real Banach space with norm $\|.\|$, X^* be its topological dual and $\langle x^*, x\rangle$ be the duality pairing between $x \in X$ and $x^* \in X^*$. We denote the closed segment $[x, y] = \{\lambda x + (1-\lambda)y \colon \lambda \in [0,1]\}$ for $x, y \in X$, and define $(x, y]$, $[x, y)$ and (x, y) similarly.

Definition 2.1 [33] Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be an extended real valued function, the effective domain is defined by

$$\text{dom}(f) = \{x \in X : f(x) < +\infty\}.$$

Definition 2.2 [5] A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be lower semi-continuous at $x \in X$ if and only if: $\forall \lambda \in \mathbb{R}$, such that $\lambda < f(x), \exists V \subset U(x) \colon \lambda < f(y) \forall y \in V$.

Definition 2.3 [3, 29] A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be quasi-convex, if for any $x, y \in X$ and $z \in [x, y]$ we have

$$f(z) \leq \max\{f(x), f(y)\}. \tag{2}$$

Definition 2.4 (See [3, 21, 29]) A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be strictly quasi-convex, if the inequality (2) is strict when $x \neq y$.

Definition 2.5 (See [12]) Let $T: X \to X^*$ be a multivalued operator with domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. T is said to be quasi-monotone if for any $x, y \in X$, $x^* \in T$ and $y^* \in T(y)$, we have

$$\langle x^*, y - x\rangle > 0 \Longrightarrow \langle y^*, y - x\rangle \geq 0.$$

Definition 2.6 (See [12]) Let $T: X \to X^*$ be a multivalued operator with domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. T is said to be pseudo-monotone if for any $x, y \in X$, $x^* \in T$ and $y^* \in T(y)$, we have

$$\langle x^*, y - x\rangle \geq 0 \Longrightarrow \langle y^*, y - x\rangle \geq 0. \tag{3}$$

Definition 2.7 (See [29]) Let $T: X \to X^*$ be a multivalued operator with domain $D(T) = \{x \in X : T(x) \neq \emptyset\}$. T is said to be strictly pseudo-monotone if for any different two points $x, y \in X$, $x^* \in T$ and $y^* \in T(y)$, we have

$$\langle x^*, y - x\rangle \geq 0 \Longrightarrow \langle y^*, y - x\rangle > 0. \tag{4}$$

Definition 2.8 (See [2]) An operator ∂ that associates to any lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$

and a point $x \in X$ a subset $\partial f(x)$ of X^* is a sub-differential if it satisfies the following properties:

(i) $\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle + f(x) \le f(y), \forall y \in X\}$, whenever f is convex;

(ii) $0 \in \partial f(x)$, whenever $x \in dom\, f$ is a local minimum of f;

(iii) $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$, whenever g is a real a real-valued convex continuous function which is ∂-differentiable at x,

where g-differentiable at x means that both $\partial g(x)$ and $\partial(-g)(x)$ are non-empty. We say that f is ∂-differentiable at x when $\partial f(x)$ is non-empty while $\partial f(x)$ are called the sub-gradients of f at x.

Definition 2.9 (See [2]) The Clarke-Rockafellar generalized directional derivative of f at $x_0 \in \mathrm{dom}(f)$ in the direction $d \in X$ is given by

$$f^{\uparrow}(x_0, d) = \sup_{\varepsilon>0} \operatorname{limsup}_{\substack{x \to_f x_0 \\ \lambda \searrow 0}} \inf_{d' \in B_\varepsilon(d)} \frac{f(x+\lambda d') - f(x)}{\lambda}, \tag{5}$$

where $B_\varepsilon(d) = \{d' \in X : \|d' - d\| < \varepsilon\}$, $\lambda \searrow 0$ indicates the fact that $\lambda > 0$ and $\lambda \to 0$,
and $x \to f^{x_0}$ means that both $x \to x_0$ and $f(x) \to f(x_0)$;
While,

Definition 2.10 (See [3]) The Clarke-Rockafellarsubdifferential of f at x_0 is defined by

$$\partial f(x_0) = \{x^* \in X^* : (x^*, d) \le f^{\uparrow}(x_0, d), \forall d \in X\}; \tag{6}$$

if $x_0 \in X \backslash \mathrm{dom}(f)$, then

$$\partial f(x_0) = \emptyset, ([29]).$$

Definition 2.11 (See [12, 29]) A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be quasi-convex (with respect to Clarke-RockerfellerSubdifferentials) if for any $x, y \in X$,

$$\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle > 0 \Longrightarrow \forall z \in [x, y],\ f(z) \le f(y). \tag{7}$$

Definition 2.12 (See[12, 29]) A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be pseudo-convex (with respect to Clarke-Rockerfeller Subdifferentials) if for any $x, y \in X$:

$$\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle \ge 0 \Longrightarrow f(x) \le f(y). \tag{8}$$

Definition 2.13 (See [3, 21, 29]) A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be strictly pseudo-convex (with respect to Clarke-Rockerfeller Subdifferentials) if for any two different points $x, y \in X$:

$$\exists x^* \in \partial f(x) : \langle x^*, y - x\rangle \ge 0 \Longrightarrow f(x) < f(y), \text{ when } x \ne y. \tag{9}$$

Definition 2.14 [29] A lower semi-continuous function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be radially continuous if for all $x, y \in X$, f is continuous on $[x, y]$.

Definition 2.15 [29] A function $f: X \to \mathbb{R} \cup \{+\infty\}$ is said to be radially non-constant if for all $x, y \in X$, with $x \ne y$, $f \not\equiv$ constant on $[x, y]$.

Definition 2.16 A sub-differential operator ∂f is said to satisfy the Stampacchia variational inequality (10) at $x \in C$ if:

$$\langle x^*, y - x\rangle \ge 0 \forall y \in C, \forall x^* \in \partial f(x). \tag{10}$$

Definition 2.17 While a sub-differential operator ∂f satisfies the Minty variational inequality (11) at x, if:
$\forall y \in C$, $\langle y^*, y - x\rangle \leq 0$, $\forall y^* \in \partial f(y)$.

(11)
Theorem 2.1 **(**Approximate mean value inequality). Let$f: X \to \mathbb{R} \cup \{+\infty\}$ be a Clarke-Rockafeller sub-differentiable lower semi-continuous (l.s.c.) function on a Banach space X. Let$a, b \in X$ with $a \in$ dom f and $a \neq b$. Let $\rho \in \mathbb{R}$ be such that $\rho \leq f(b)$. Then, there exist $c \in [a, b)$ and $x_n \to f^C$ and $x_n^* \in \partial f(x_n)$ such that

(i) $\liminf_{n\to+\infty}\langle x_n^*, c - x_n\rangle \geq 0$;

(ii) $\liminf_{n\to+\infty}\langle x_n^*, b - a\rangle \geq \rho - f(a)$.

Proof. (**See** [2]).
Lemma 2.2 Let$f: X \to \mathbb{R} \cup \{+\infty\}$ be a Clarke-Rockafeller sub-differentiable lower semi-continuous (l.s.c.) function on a Banach space X. Let$a, b \in X$ with $f(a) < f(b)$. Then, there exist $c \in [a, b)$, and two sequences $c_n \to c$, and $c_n^* \in \partial f(c_n)$ with

$\langle c_n^*, x - c_n\rangle > 0$ for every $x = c + \lambda(b - a)$ with $\lambda > 0$.

Proof. By **Theorem 2.1,** there exists an $x_0 \in [a, b)$ and a sequence $x_n \to f^C$ and $x_n^* \in \partial f(x_n)$ verifying

$\liminf_{n\to+\infty}\langle x_n^*, c - x_n\rangle \geq 0$ and $\liminf_{n\to+\infty}\langle x_n^*, b - a\rangle > 0$.

(12)
Putting $x = c + \lambda(b - a)$ with $\lambda > 0$ it holds

$\langle x_n^*, x - x_n\rangle = \langle x_n^*, c - x_n\rangle + \lambda\langle x_n^*, b - a\rangle > 0$

(13)
for n very large.

3. Relationship between Generalized Convexity and Generalized Monotonicity

Theorem 3.1 Let$f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) Clarke-Rockafeller subdifferentiable function on a Banach space X. Then, f is quasi-convex if and only if ∂f is quasi-monotone.
Proof. We show that if f is not quasi-convex, then ∂f is not quasi-monotone.
Suppose that there exist some x, y, z in X with $z \in [x, y]$ and $f(z) > \max\{f(x), f(y)\}$. According to Lemma 2.2 applied with $a = x$ and $b = z$, there exists a sequence $y_n \in \mathrm{dom}\partial f$ and $y_n^* \in \partial f(y_n)$ such that

$y_n \to \bar{y} \in [x, z]$, $\bar{y} \neq z$ and $\langle y_n^*, y - y_n\rangle > 0$. (14)

Let $0 < \lambda \leq 1$ be such that $z = \bar{y} + \lambda(y - \bar{y})$ and set $z_n = y_n + \lambda(y - y_n)$, so that $z_n \to z$. Since f is lower semi-continuous, we may pick $n \in \mathbb{N}$ very large with $f(z_n) > f(y)$. Apply Lemma 2.2 again with $a = y$ and $b = z_n$ to find sequences $x_k \in \mathrm{dom}\partial f$, $x_k^* \in \partial f(x_k)$ such that

$x_k \to \bar{x} \in [y, z_n]$, $\bar{x} \neq z_n$ and $\langle x_k^*, y_n - x_k\rangle > 0$. (15)

In particular, $\bar{x} \neq y_n$ and

$\langle y_n^*, \bar{x} - y_n\rangle = \frac{\|\bar{x} - y_n\|}{\|y - y_n\|}\langle y_n^*, y - y_n\rangle > 0$; (16)

hence, $\langle y_n^*, x_k - y_n\rangle > 0$ for k sufficiently large. But $\langle y_n^*, y_n - x_k\rangle > 0$, showing that ∂f is not quasi-monotone.

Conversely, we suppose that f is quasi-convex and show that ∂f is quasi-monotone. Let $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ with $\langle x^*, y - x\rangle > 0$. We need to verify that $f^\uparrow(y, x - y) \leq 0$. We fix $\varepsilon > 0$ and $\omega \in (0, \varepsilon)$ such that

$\langle x^*, v - x\rangle > 0$ for all $v \in B_\omega(y)$.

We fix $v \in B_\omega(y)$. Since $f^\uparrow(y, x - y) > 0$ we can find $\varepsilon' \in (0, \varepsilon - \omega)$, $u \in B_{\varepsilon'}(x)$ and $t \in (0,1)$ such that

$f(u+t(v-u)>f(u)$. From the quasi-convexity of f we deduce that $f(u)<f(v)$, whence,

$$f(v+\lambda(u-v))\leq f(v) \text{ for all } \lambda\in(0,1),$$

so that

$$\inf_{\mu\in B_\varepsilon(x-y)}\frac{f(v+\lambda\mu)-f(v)}{\lambda}\leq\frac{f(v+\lambda(u-v))-f(v)}{\lambda}\leq 0 \text{ for all } \lambda\in(0,1).$$

Combining the inequalities and for any $\varepsilon>0$ there exists $\omega>0$ such that

$$\sup_{\substack{v\in B_\omega(y)\\ \lambda\in(0,1)}}\left[\inf_{\mu\in B_\varepsilon(x-y)}\frac{f(v+\lambda\mu)-f(v)}{\lambda}\right]\leq 0,$$

which shows that $f^{\uparrow}(y,x-y)\leq 0$.

We consider the relationship between pseudo-convexity and quasi-convexity.

Theorem 3.2 Let $f\colon X\to\mathbb{R}\cup\{+\infty\}$ be a lower semi-continuous (l.s.c.) function on a Banach space X such that f Clarke-Rockafeller suddifferentiable. Consider the following assertions:

(i) f is pseudoconvex.

(ii) f is quasiconvex and ($0\in\partial f(x)\Longrightarrow x$ is a global minimum of f).

Then, (i) implies (ii). And (ii) implies (i) if f is radially continuous.

Proof. (i) ⟹(ii). We want to prove that f is quasiconvex. Suppose to the contrary that for some $x,y\in X$, $z\in(x,y)$ we have $f(z)>\max\{f(x),f(y)\}$. Since f is lower semicontinuous, we can find some $\varepsilon>0$ such that $f(z')>\max\{f(x),f(y)\}$, for all $z'\in B_\varepsilon(z)$. Since z cannot be a local nor global minimizers, there exist some $v\in B_\varepsilon(z)$ such that $f(v)<f(z)$. From **Lemma 2.2**, there exist $u_n\to u\in[v,z)$ and $u_n^*\in\partial f(u_n^*)$ such that

$$\langle u_n^*,z-u_n\rangle>0.$$

But since $z\in(x,y)$, either of the following must hold

$$\langle u_n^*,x-u_n\rangle>0 \text{ or } \langle u_n^*,y-u_n\rangle>0.$$

Therefore,

$$f(u_n)\leq\max\{f(x),f(y)\}.$$

Which is a contradiction.

(ii) ⟹(i). Let $x\in\operatorname{dom}\partial f$, $y\in X$, and $x^*\in\partial f(x)$ such that $\langle x^*,y-x\rangle\geq 0$. If $0\in\partial f(x)$, then x is a global minimum of f and $f(x)\leq f(y)$ in particular. Otherwise, $[0\notin\partial f(x)]$, there exist $d\in X$ such that $\langle x^*,d\rangle>0$. We define a sequence $\{y_n\}$ by

$$y_n=y+\left(\frac{1}{2n\|d\|}\right)d.$$

For every $n\in\mathbb{N}$, the point y_n satisfies

$$y_n\in B_{1/n}(y),$$

$$\langle x^*,y_n-x\rangle=\langle x^*,y_n-y\rangle+\langle x^*,y-x\rangle\geq\left(\frac{1}{2n\|d\|}\right)\langle x^*,d\rangle>0.$$

Using (8), we obtain that, for every n,
$f(y_n)\geq f(x)$ and by radial continuity of f, $f(y)\geq f(x)$.

Theorem 3.3 Let$f\colon X\to\mathbb{R}\cup\{+\infty\}$ be a lower semi-continuous (l.s.c.) Clarke-Rockafeller subdifferentiable function. Consider the following assertions:

(i) f is pseudoconvex.

(ii) ∂f is pseudomonotone

Then, (i) implies (ii). And (ii) implies (i) if f is radially continuous.

Proof. (i) ⟹(ii). Suppose $x^*\in\partial f(x)$ such that $\langle x^*,y-x\rangle\geq 0$. By **Theorem 3.2,** f is quasiconvex. By

Theorem 3.1, we conclude that ∂f is quasimonotone. Hence $\langle y^*, y - x\rangle \geq 0$, for all $y^* \in \partial f(y)$. Suppose to the contrary that for some $y^* \in \partial f(y)$, we have $\langle y^*, y - x\rangle = 0$. From (8), we obtain $f(x) \geq f(y)$.

However, since $f^{\uparrow}(x, y - x) > 0$, there exist $\varepsilon > 0$, such that for some $x_n \to x$, $\lambda_n \searrow 0$ and for all $y' \in B_\varepsilon(y)$, we have $f\big(x_n + t_n(y' - x_n)\big) > f(x_n)$. By the quasiconvexity of f, it implies that $f(y') > f(x_n)$ for every $y' \in B_\varepsilon(y)$. In particular, $f(y') > f(x)$ because f is lower semicontinuous. Thus, $f(y') \geq f(y)$. This shows that y is a local minimum and also a global minimum, which is a contradiction since we can have that $f(y) > f(x_n)$.

(ii) $\Longrightarrow$(i). Using Theorem **,** we prove that f is pseudoconvex. Since ∂f is pseudomonotone, ∂f is quasimonotone. By Theorem 3.2, f is quasi-convex. On the other hand, if x is not a minimizer of f, there exists $y \in X$ such that $f(y) < f(x)$. Using **Lemma 2.2,** we find $u \in \mathrm{dom}\partial f$ and $u^* \in \partial f(u)$ such that $\langle u^*, x - u\rangle > 0$ and by the pseudo-monotonicity of ∂f, $\langle x^*, x - u\rangle > 0$ for every $x^* \in \partial f(x)$. Hence, 0 does not $\partial f(x)$. Consequently, f satisfies condition $0 \in \partial f(x)$, which implies that x is a global minimum of f, which completes the proof. ■

Theorem 3.4 Let$f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) Clarke-Rockafeller subdifferentiable function on a Banach space X. Consider the following assertions:

(i) f is strictly pseudoconvex.

(ii) f is strictly quasiconvex and ($0 \in \partial f(x) \Longrightarrow x$ is a global minimum of f),

Then, (i) implies (ii). And (ii) implies (i) if f is radially continuous.

Proof. (i) $\Longrightarrow$(ii). We want to prove that f is strictly quasiconvex. Let $f$be a strictly pseudo-convex function, then by **Theorem 3.2,** the function f is quasiconvex and satisfies the optimality condition

$0 \in \partial f(x) \Longrightarrow$ (x is a global minimum of f).

Since f is quasiconvex, then according to [14], it suffices to prove that $f$is radially non-constant. Assume by contradiction that thereexists a closed segment $[x, y]$ with with $x \neq y$ where with f is constant. Let $z \in (x, y)$and apply the strict pseudo-convexityproperty to x and z, then

$f(z) = f(x) \Longrightarrow (\forall z^* \in \partial f(z){:}\langle z^*, x - z\rangle < 0)$.

Using the same argument for $z$and $y$we obtain

$f(z) = f(y) \Longrightarrow (\forall z^* \in \partial f(z){:}\langle z^*, y - z\rangle < 0)$.

Since $\partial f(z)$ is nonempty, it follows that
for all $z^* \in \partial f(z)$,$\langle z^*, x - y\rangle < 0$and $\langle z^*, x - y\rangle > 0$), which is a contradiction.

(ii) $\Longrightarrow$(i). Assume that $f$satisfies condition ii) and f is radially continuous. Then by Theorem 3.2, f is pseudoconvex. We prove that $f$is pseudo-convex. Suppose by contradiction that there exist $x \neq y$ in X and $x^* \in \partial f(x)$ such that

$\langle x^*, y - x\rangle \geq 0$ and $f(x) \geq f(y)$.

Then, it follows by pseudo-convexity property that

$\forall z \in [x, y]$, $f(z) = f(x)$.

Since $f$is quasi-convex, then we have

$\forall z \in [x, y]$, $f(z) \geq f(x) \geq f(y)$.

So $f$is not radially non-constant on X (since $f$is constant on $[x, y]$) which contradicts the fact $f$is strictly quasi-convex.

Theorem 3.5 Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous (l.s.c.) function such that f is radially Clarke-Rockafeller Sub-differentiable. Consider the following assertions:

(i) f is strictly pseudo-convex.

(ii) ∂f is strictly pseudomonotone

Then, (i) implies (ii). And (ii) implies (i) if f is radially continuous.

Proof. (i) ⟹(ii). Suppose that f is strictly pseudoconvex. We want to prove that ∂f is strictly pseudomonotone. Suppose to the contrary that there exist two distinct points $x, y \in X$, $x^* \in \partial f(x)$ and $y^* \in \partial f(y)$ such that $\langle x^*, y - x\rangle \geq 0$ and $\langle y^*, y - x\rangle \leq 0$.
Since f is strictly pseudoconvex, we have that
$f(x) < f(y)$ and $f(y) < f(x)$.
Which is a contradiction. Therefore, ∂f is strictly pseudomonotone.

(ii) ⟹(i). Suppose that f satisfies condition (ii) and f is radially continuous. We want to prove that f is strictly pseudoconvex. Suppose to the contrary that there exist two distinct points $x, y \in X$, and $x^* \in \partial f(x)$ such that

$\langle x^*, y - x\rangle \geq 0$ and $f(x) \geq f(y)$.

Then,

$\langle x^*, z - x\rangle \geq 0$ for all $z \in [x, y]$. (17)

By theorem 3.3, f is quasiconvex. Consequently, f must be constant on $[x, y]$. Contrarily, from (16) and the strict monotonicity of $\partial f(x)$, we have

$\langle x^*, z - x\rangle > 0, \forall z \in (x, y)$ and $\forall z^* \in \partial f(z)$. (18)

Pick $z_0 \in (x, y)$ such that $\partial f(z_0) \neq \emptyset$ (such a z_0 exists since f is a radially Clarke-Rockafeller sub differentiable function). Choose any $z_0^* \in \partial f(z_0)$. Then, $\langle z_0^*, z_0 - x\rangle > 0$. Therefore, $\langle z_0^*, y - z_0\rangle > 0$. Consequently, there exist $\varepsilon > 0$ such that

$\langle z_0^*, y' - z_0\rangle > 0$ for all $y' \in B_\varepsilon(y)$.

By the pseudo-convexity of f, it follows that y is a global minimum of f. Hence, z_0 is also a global minimum of f. Thus, $0 \in \partial f(z_0)$ and this is a contradiction with (19).

4. Optimality Conditions and Variational Inequalities

We study the necessary and sufficient conditions for a point x to be a global minima of a pseudo-convex, lower semi-continuous and radially continuous function f over a convex set C.

Theorem 4.1 Let$f: C \subseteq X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous Clarke-Rockafeller differentiable pseudo-convex function, and $\bar{x} \in C$. Then the following assertions are equivalent

(i) $\bar{x}$ is an optimal solution of (1).

(ii) ∂f satisfies (11) at $\bar{x} \in C$.

Proof. (i) ⟹(ii). Suppose that $\bar{x}$ is a solution of (1), then by Definition 2.12, if $f(\bar{x}) \leq f(x)$, then we have

$\forall x^* \in \partial f(x), \qquad \langle x^*, \bar{x} - x\rangle \leq 0.$

So, the variational inequality (11) holds.

(ii) ⟹(i). let $x \in C$ such that $x \neq \bar{x}$ then for some $y \in (\bar{x}, x)$. Then,

$\forall y^* \in \partial f(y), \qquad \langle y^*, \bar{x} - y\rangle \leq 0.$

Which follows that

$\forall y^* \in \partial f(y), \qquad \langle y^*, x - y\rangle \leq 0.$

Since $\partial f(y)$ is nonempty and from the pseudoconvexity of f we have

$\forall y \in (\bar{x}, x), \qquad f(y) \leq f(x).$

But since f is lsc, then $f(\bar{x}) \leq f(x)$. ■

We proceed to maximization problem.

Let $C \subseteq X$ be convex and nonempty. Consider the maximization problem:

$maximise\ f(x)$, subject to $x \in C$, (19)

where fa strictly pseudoconvex lower semi-continuous (l.s.c.) and radially Clarke-Rockafeller differentiable function.

Theorem 4.3 Let $f: C \subseteq X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous Clarke-Rockafeller Sub-differentiable pseudo-convex function. let $\bar{x} \in C$ such that

$$-\infty \leq \inf_C f < f(\bar{x}). \quad (20)$$

Then, $\bar{x}$ is a maximum of f on C if and only if for all $x \in C$ such that $f(x) = f(\bar{x})$ and for all $x^* \in \partial f(x)$ we have:

$$\langle x^*, y - x \rangle \leq 0 \quad \forall y \in C \backslash \{x\}. \quad (21)$$

Proof.$\Rightarrow$Suppose that x is a solution of (14). Let $x \in X$ such that $f(x) = f(\bar{x})$ and let $x^* \in \partial f(x)$. Then,

$$f(y) = f(x), \forall y \in C.$$

Since f is strictly pseudoconvex, then

$$\langle x^*, y - x \rangle < 0, \forall y \in C \backslash \{x\}.$$

$\Leftarrow$ Suppose that there exist $z \in C$ such that $f(z) > f(\bar{x})$. By the hypothesis, there exist $z_0 \in C$ such that $f(z_0) < f(\bar{x})$. Since f is strongly radially Clarke-Rockafeller sub-differentiable, then there exists some $x_0 \in (z_0, z)$ such that $f(x_0) = f(\bar{x})$ and $\partial f(x_0) \neq \emptyset$. Pick any $x_0^* \in \partial f(x_0)$. Then,

$\langle x_0^*, z - x_0 \rangle < 0$ and $\langle x_0^*, x_0 - x_0 \rangle < 0$, which is a contradiction.

Thus, $\bar{x}$ is a maximum of f on C.

5. Conclusion

We extended the relationships between convex functions and corresponding monotone maps to pseudo-convexity and the corresponding pseudo-monotonicity of their sub-differentiable maps. We characterized the lower semi-continuous Clarke-Rockafeller sub-differentiable pseudo-convex functions (1) by the corresponding monotonicity of their Clarke-Rockafeller sub-differential operator ∂f and presented their optimality conditions, using variational inequalities.

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