A Mathematical Literature of Nested Intervals from "Archimedes" to "Georg Ferdinand Ludwing Phillip Cantor"

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Abstract: The concept of convergent sequences, which is similar to the theory of nested intervals, has been around since antiquity. Archimedes used two sets of values, ambient and nested, to approximate the unknown in excess and deficiency. The idea of a point sitting inside a series of nested intervals was developed by Jean Buridan. Pierre de Fermat, Derek Gregory, Issac Newton, Colin MacLaurin, Carl Friedrich Gauss, and Jean-Baptiste Joseph Fourier employed excess and deficiency approximations to find an unknown value. This logical structure evolved into the analysis argumentation technique in the works of Bernard Bolzano, Augustin-Louis Cauchy, Johann Peter Gustav Lejeune Dirichlet, Karl Weierstrass, and Georg Ferdinand Ludwing Phillip Cantor in the 19th century. In the 1870s, Charles Méray, Weierstrass, Heinrich Eduard Heine, Cantor, and Richard Dedekind developed the idea of a real number. Cantor's development was predicated on the idea of a limiting point and the nested interval theory. We will now examine the origins of this concept, which may be traced back to ancient world.

Keywords: Convergent sequences, Nested intervals, Archimedes, Mathematicians (Mac Laurin, Cantor, Kolmogorov, etc.), History

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1 | Introduction

In the latter part of the 1800s, the idea that a collection of real numbers is continuous or complete was expressed in a variety of ways. Each of these ideas was based on the following characteristics of a set of real numbers:

- Each sequence of closed nested intervals has a nonempty intersection, and Archimedes' axiom is valid.

- Each bounded subset has an upper bound.
- Each Cauchy sequence converges, and Archimedes' axiom is valid.
- Each infinite bounded subset has a limiting point (Bolzano-Weierstrass property).
- Each bounded monotonic sequence converges.

At the beginning of the 20th century, Hilbert demonstrated that both qualities were equivalent. All of the ideas came from antiquated techniques of fatigue and proportionality. But it took over two thousand three hundred years for the ideas of a number and continuity to emerge. The history of this process is rich and interesting, and it keeps a lot of pages open. We are only going to unravel the first of the ideas here, i.e. the method of nested intervals or, which makes about as much sense, convergent sequences.

2 | Archimedes

The first theory of a real number was explained by Eudoxus, set forth by Euclid in his Elements, and consisted of two parts: the theory of proportions and the exhaustion method which was elaborated for incommensurable values and involved elaboration of a monotonic sequence of sums of known values approximating deficiency to the sought-for geometrical value. In his "correspondence with Dositheus" cycle ('The Quadrature of the Parabola, 'On the Sphere and

Fig. 2.1 Archimedes

The Cylinder', 'On Conoids and Spheroids, 'On Spirals', and 'On the Measurement of a Circle'),in order to calculate the sought-for value, Archimedes created two sequences of values measured which approximates to the sought-for value in excess and deficiency. "All these works were written in the form of letters to Dositheus of Pelusium, pupil of Conon of Samos".

In his letter to Eratosthenes (The Method of Mechanical Theorems), Archimedes noted that the exhaustion method was convenient when it was necessary to demonstrate the correctness of the foregone conclusion. In order to find the conclusion itself, Archimedes used the heuristic mechanical method of mathematical atomism. He presented intervals of lines consisting of material points, planar figures consisting of intervals, and bodies consisting of flats, and he determined distances between centers of gravity.

In his work 'On the Sphere and the Cylinder' Archimedes showed that the ratio of sums of two sequences was nearer and yet nearer to the unity. In the same work, Archimedes introduced the fifth assumption which was later named Archimedes axiom: "The larger of two unequal lines, surfaces or bodies is larger than the smaller one by a value which, if added to itself, can exceed any given value of those which may be in certain relation with one another" –it is translation from Russian [1]. In Health translation: "Of unequal lines, unequal surfaces and unequal solids, the greater exceed the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with one another" [2]. Alternatively, the difference between partial sums of both sequences can be made arbitrary small.

3 | During 12th Century and 13th Century

In universities of Paris of the 12th century and those of Oxford of the 13th century, lectures of professors involved reading classics, particularly Aristotle, and commenting upon them. In 1958, as V.P. Zubov noted, the "mathematization" of Aristotle and "physicalization" of Euclid were characteristic of the 14th century [3]. The concepts of a point, line, and surface were separated from their physical meaning by scientists. The idea of a sequence and infinite successive sequence appeared in works of Averroes and Albert the Great as a movement characteristic; and the difference between the kinematic and dynamic aspects was conceived. During developing the works of the Calculators, a group of scientists from Merton College in Oxford, the ancient tradition of assessment with the help of inequations was replaced by a new tradition of exact calculation, i.e. the equation. It was their merit that the notions of a "sequence", "intensity", and "instantaneous velocity" were introduced in the science, although not precisely defined. In 1346, Richard Swineshead first discussed physical notions of a change and movement in mathematics, intension and remission of qualities (density and tenuity, force and resistance, quickness and slowness, warmth and coldness) [4].

4 | Jean Buridan

Figure 4.1, Jean Buridan

Jean Burdian (about 1301-1362) was a French priest, philosopher and scientist who taught in the University of Paris. Buridan highlighted the concept of a bound and highlighted the significance of the geometrical concept of “contingence”. V.S. Shirokov observed the connection between this idea and the concept of "contingence" in Lobachevsky's writings [5]. Like Thomas Bradwardine, Buridan thought that although the continuum was made up of points, its “point or instant were infinitely small” [6]. Ernst Zermelo (a German logician and mathematician) examined the problem of ordering and choice in 1904.

Buridan developed a framework that provided the foundation for a significant 19th-century mathematical advancement: a sequence of intervals each of which contained a continuum point. Bolzano proposed the covering notion in 1817; it was included in Dirichlet's lectures in 1862; Heine stated the covering lemma in 1872; Borel proved the covering theorem in 1895; and Lebesgue expanded it in 1898.

5 | Throughout 16th century and 17th century

Interest in the Archimedes legacy has increased since the Renaissance. His book "Measurement of the Circle," which was translated from Arabic into Latin early in the 12th century, was well-known in Europe. In 1269, William of Moerbeke, an Albert the Great pupil, translated everything important Greek writings byArchimedes. Archimedes' writings were translated into Latin and published in Bazel in Greek in 1544, which helped spread his ideas throughout. His argument was replicated by scientists in both new and old issues. ‘Archimedes’ works had a great influence on Galilee and Stevin. One could come across the method of elaboration of sequences of ambient and nested values in interpolation of high order differences in ‘Logarithmic Arithmetic’ by G. Briggs in 1624 [7], in P. Fermat’s ‘Méthodes de quadrature’ [8]. P. Fermat told about them to a student of Galilee, B. Cavalieri; in interpolation formulas of D. Wallis, calculating number $\frac{4}{\pi}$ in 1656 [9].

6 | Both Johannes Kepler and Bonaventura Cavalieri: Mathematical atomism, geometrical algebra as a heuristic method

Figure 6.1: Johannes Kepler (**1571 –1630**)

Figure 6.2: Bonaventura Cavalieri (**1598-1647**)

Both Johannes Kepler and Bonaventura Cavalieri employed methods related to infinitesimal quantities (indivisibles/mathematical atomism) as a heuristic method to determine areas and volumes, which were significant precursors to the formal development of integral calculus. While they did not use modern "geometrical algebra," their work involved imaginative geometric approaches to problem-solving.

Johannes Kepler

Kepler employed infinitesimally small geometric quantities in his work, particularly in his 1615 book Stereometria doliorum (Solid Geometry of Wine Barrels).

Mathematical Atomism:- He conceived of solids as being composed of an infinite number of infinitesimally thin layers (or "indivisibles"). He used this concept to calculate the volumes of various containers, essentially anticipating integral calculus techniques.

Heuristic Method:- His approach was practical and intuitive rather than rigorously proven by Euclidean standards. It was part of his broader effort to establish a physical and mathematical basis for the universe, which led to his laws of planetary motion. His methods were highly influential and directly spurred Cavalieri's more systematic development of the idea.

Bonaventura Cavalieri

Cavalieri, a student of Galileo, developed Kepler's ideas into a more systematic, albeit not fully rigorous, method.

Mathematical Atomism (Method of Indivisibles):- In his 1635 work Geometria indivisibilibus continuorum, Cavalieri posited that a planar figure was made up of an infinite, parallel set of lines ("indivisibles"), and a solid was made up of an infinite set of parallel planar areas. His famous "Cavalieri's Principle" states that if two solids have the same cross-sectional area at every height, then they have the same volume.

Heuristic Method/Geometrical Algebra:- The method allowed for the rapid and simple calculation of areas and volumes for complex figures, effectively integrating x^n . While powerful, it faced criticism for a lack of rigor, as comparing infinities was controversial at the time. Cavalieri himself acknowledged that rigor was a concern of philosophy, not necessarily of geometry in this innovative context. His work provided the crucial foundation for the formal development of integral calculus by Newton and Leibniz.

As a consequence, the two approaches put forward in Archimedes' writings have taken shape: searching for the intended outcome using the indivisibles technique, which was created in Kepler and Cavalieri's works, and reasoning about the outcome that was discovered. With the aid of sequences that converged in excess and deficiency, which J. Gregory and C. MacLaurin advanced. Finding a geometrical value (length, area, or volume) was the aim of problem solving; in the 17th century, this method did not offer a definition of a number as such.

7 | James Gregory: “The True Areas of the Circle and the Hyperbola”

Figure 7.1 James Gregory

James Gregory (November 1638 – October 1675) was a Scottish mathematician and astronomer. he published his two works: ‘The True Areas of the Circle and the Hyperbola’ [10] and ‘General Sections of Geometry’ [11] where he applied the method of Archimedes to find areas of curved surfaces, however, as he himself noted, in combination with a more convenient and brief method of indivisibles proposed by Cavalieri. Gregory’s ‘Geometry’ already contained a proportion which enabled finding the length of a curve with the help of an element of the curve and the main ideas of integral calculus. The term “convergence” was originally used by Gregory. Gregory produced sequences that approximated the real value of the area of the hyperbolic segment in excess and deficiency in the same year's work, "The True Areas of the Circle and the Hyperbola" [10], which expressed all relations in proportions of inscribed and circumscribed figures. Hence, the tradition of Archimedes took another twist based on the method of indivisibles, which enabled a simplified work with proportional quantities (areas).

8 | Isaac Newton

Figure: 8.1 Isaac Newton

The quadrature computation in Newton's interpolation formulae is based on the Archimedes tradition. In 1669, he created an approximate method for algebraic equations using tangents [12]. Thomas Simpson enhanced this technique in 1740 [13]; in 1768, J. Mourraille (J.-R.-P. Mourraille, 1720-1808) noted that the convexity must be in place [14]; in 1826, Jean-Baptiste Joseph Fourier described the convergence conditions of this

method [15]. Using this technique, the interval that contained the equation's root was contracted. In the mid-1900s, when Stefen Banach established the concept of contracting maps in 1922 [16], the L.V. Cantorovich expanded the tangent approach in his works based on it (e.g. in [17]).

9 | Colin Mac Laurin

Fig. 9.1: Colin MacLaurin

"A Treatise of Fluxions", published in 1742 by Scottish mathematician Colin Maclaurin, was a foundational two-volume work that rigorously established Newton's calculus (fluxions) using "Archimedean geometry", providing a defense against " Bishop Berkeley's critiques" and laying groundwork for modern analysis, despite limited immediate teaching influence. Colin MacLaurin (1698-1746) was a friend of Newton on whose recommendation he became professor of Edinburgh University, having in 1726 replaced James Gregory Jr. in his position. James Gregory (1666-1742) was the professor of Mathematics and nephew of a prominent mathematician James Gregory (1738-1675), and brother of an astronomer and mathematician David Gregory (1659-1708). Newton's theory of analysis was criticized for its lack of logic and clarity after his death in 1727. "The Analyst," written by J. Berkeley and published in 1734, was the pinnacle of this critique. MacLaurin decided to write his reasoning for Newton's Treatise of Fluxions. In 1742, his Treatise of Fluxions was published [18]. It included a methodical and clear statement of Newton's method. This work was intended to be a textbook for youth. MacLaurin was trying to demonstrate the closeness of Newton's method and the ancient exhaustion method. An experienced educationalist, he found a perceivable image of convergence, having arranged convergent values – whether lengths, areas, or volumes – on a straight line. MacLaurin was the first to introduce the term ‘Archimedes’ axiom’.

10 | During 18th Century

The 18th century saw the continuation of the finite difference relations method tradition, which began with the writings of B. Taylor, A. de Moivre, and I. Newton and was developed by J. Stirling, L. Euler, and J.L. Lagrange. A.-M. Legendre characterized each geometrical value as a number in his 1794 "Elements of Geometry" [19] and discovered a geometrical value that matched each number. Notable is the development of Ampère's 1806 proof of the Lagrange mean value theorem. He created ongoing inequalities using the calculated relation (mean fraction) in the middle and approximated the desired value by lowering the pitch [20]. Ampère's work had neither geometric connections nor illustrations. Lagrange and Laplace developed the response error valuation issue that Galilee had already established. Lagrange's "Memoir on applying the method of averaging out results of a large number of observations where advantages of this method for calculation of probabilities are considered and where various problems related to this issue are solved" [21] was published in 1775/1776. Lagrange examined the likelihood of a mistake in the arithmetic average in several rules of error allocation in this memoir.
This Gauss conclusion is referred to be the earliest formulation of the squeezing theorem in Western reference literature. In academic folklore, this theorem is referred to as the "squeezed theorem," "pinched theorem," "sandwich theorem," "three chords," "two militiamen," "carabineers," "gendarmes," "police," and "a drunkard and two policemen." It is typically distinguished as an independent theorem in 20th-century analytic textbooks. They needed the notion of a converging sequence and continuous function for this purpose.

11 | Bernard Bolzano

Fig. 11.1 Bernard Bolzano

In 1817, the work of Bernard Bolzano entitled “Purely analytic proof of the theorem that between any two values which give results of opposite sign, there lies at least one real

root of the equation" [22], Bernard Bolzano also introduced the convergence test of a sequence of partial sums. The the idea of Archimedes, Bolzano demonstrated that if the difference between partial sums (elements) of the sequence could be made arbitrarily small, then each such
sequence would converge to a certain limit. Cauchy repeated this criterion without
any proof in 1821. Since then, it has been called the Cauchy criterion, and
sequences which meet this criterion were thereafter called Cauchy-Cantor
sequences. Bolzano's initial contribution to the development of the idea of a real number was this. His second contribution made in the same work was the creation of a notion of the least upper bound of a linear numerical area (supremum). He only pointed out that the upper bound does not have to be a part of the set in question, hence it may not include the greatest element.

Bolzano intuitively established the principles of converging sequences, embedding sections, and supremum in this way, laying the groundwork for future developments in the concept of a number.

In 1830s, Bolzano started creating the concept of a real number in terms of sections [23,24]. Publication of these manuscripts did not start until 1930. Unfortunately, Bolzano's involuntary dismissal from teaching and scientific isolation prevented his works from being disseminated. Although Bolzano had brilliantly mastered mathematical technique and was the first to introduce the analytical proof in analysis, his articles were fundamental and even philosophical in their nature rather than practical. Being in advance of their time, they gained popularity but half a century later. However, it should be noted that Cauchy, without referring to Bolzano, repeated both his definition of a continuous function [25] and sequence convergence criterion with anticipatory reasoning of the geometrical progression and binomialexpansion practically word for word.

12 | Augustin-Louis Cauchy

Fig.12.1 Augustin-Louis Cauchy

In 1821 to 1823, the course of analysis given by Cauchy at École Polytechnique was brief and application-oriented. However, this course contained a systematic statement of the theory of limits, the most important analysis theorems devoted to continuous functions, differentiation,

integration, and the theory of series. At the same time, Cauchy did not address the notion of irrational numbers considering them to be limits of sequences of rational numbers without defining the operations. The genius of Augustin Cauchy (1789-1857) was in the strict and clear generalization of achievements of his predecessors. Based on these achievements, he created a harmonious concept of analysis, which enabled him to obtain new results and form new sections in mathematics, e.g. the theory of residues.
As the function is continuous, the common limit of sequences of the argument is the root of the equation. Bolzano proved this theorem in 1817 bisecting the interval and demonstrating with the help of this elaboration consistency of the existence of a supremum. As in many other cases [26], Cauchy ingeniously simplified the idea of its proof and, in effect, formalized the squeeze theorem. Currently, the theorem on the existence of a root of a continuous function is known as Bolzano-Cauchy theorem whose first time ever statement for polynomials goes back to Michele Rolle in 1690. Cauchy's reasoning was completely identical to that of Bolzano, including the preliminary resort to the geometrical progression. However, Cauchy introduced more
convenient designations, and his statement was eloquent and concise. Stated by Cauchy, ideas that had been uttered by predecessor mathematicians took a rigid and orderly form, having developed into a comprehensive and relevant (for his day) course of analysis.

13| Jean Gaston Darboux

Fig. 13.1 Jean-Gaston Darboux

Jean Gaston Darboux is a French mathematician. It is popular to define the Riemann integral as the Darboux integral. This is because the Darboux integral is technically simpler and because a function is Riemann-integrable if and only if it is Darbouxintegrable. The upper and lower integrals are the infimum and supremum, respectively, of the upper and lower sums. The Darboux integral exists if and only if the upper and lower integrals are equal [27].

14 | David Hilbert

Fig. 14.1 David Hilbert

David Hilbert was a German mathematician and philosopher of mathematics and one of the most influential mathematicians of his time. The reform and arithmetization of mathematical analysis of the 19th century reached its culmination with the creation of the concepts of a real number, continuity, and axiomatization of arithmetic [28]. Weierstrass asserted that a point on a straight line corresponded to each number. However, he was not sure whether the opposite was true. Cantor stated that there existed an unambiguous correspondence between numbers and points of a straight line but this could not be proved. With the help of the notion of a section, Dedekind demonstrated the continuity of a geometrical line and continuity of a set of real numbers [29]. The emergence of non-Euclidean geometry led to the need to analyze the axioms of geometry, the notion of continuity, and completeness. The axiomatic system of arithmetic appeared in the works of Dedekind and Peano. The systems of axioms of arithmetic and geometry had to be generalized on a common base. In 1899, Hilbert introduced a new system of axioms, having included the Archimedean and completeness axioms in it.

Several attempts were made in the 19th century to build a geometry without the Archimedean axiom: in 1890, in his 'Foundations of Geometry', G. Veronese (1854–1917) proposed the concept of linear non-Archimedean continuum [30]; works of O. Stolz [31]. D. Hilbert [32] discussed this issue. According to Hilbert, "The completeness axiom gives us nothing directly concerning the existence of limiting points, or of the idea of convergence. Nevertheless, it enables us to demonstrate Bolzano's theorem by virtue of which, for all sets of points situated upon a straight line between two definite points of the same line, there exists necessarily a point of condensation, that is to say, a limiting point. From a theoretical point of view, the value of this axiom is that it leads indirectly to the introduction of limiting points, and, hence, renders it possible to establish a one-to-one correspondence between the points of a segment and the system of real numbers. However, in what is to follow, no use will be made of the “axiom of completeness” [33].

15 | Andrey Nikolaevich Kolmogorov

Fig.15.1 Kolmogorov

During the 20th century, investigations of Kolmogorov demonstrated that the completeness axiom can be replaced by the principle of embedded sections (Cauchy-Cantor fundamental sequences) together with the Archimedean axiom [34, 35]. In the 1940s, Kolmogorov created an elaboration of real numbers as functions of anatural number [34].

Already, late in the 19 th century, new concepts of a number, continuity, and the theory of sets were included in the courses on the theory of functions of a real variable. In the 20th century, such a course was given in Russia by S.O. Shatunovsky in Odessa, by his student G.M. Fichtengolz in St. Petersburg, and by N.N. Luzin, P.S. Aleksandrov, and A.N. Kolmogorov in Moscow. The original edition of the book of Aleksandrov and Kolmogorov entitled “Introduction into the theory of functions of a real variable” was published in 1933. Neither this edition nor the following two contained any axiomatic elaboration.

According to V.M. Tikhomirov, “In the autumn of 1954, A.N. Kolmogorov started giving the course of lectures entitled ‘Analysis III’ for the third-year students of the Department of Mechanics and Mathematics at which the author quoted here in studied. This was the first synthetic (i.e. incorporating several sections of mathematics) course in the history of the Department of Mechanics and Mathematics of Moscow State University. The program of the course was developed by A.N. Kolmogorov in 1940s and 1950s.

Kolmogorov introduced the axiomatic definition of real numbers as a totality which constitutes a complete linear ordered field. Having defined algebraic relations and the relation of order, he proceeded to the last axiom, i.e. completeness axiom.

Kolmogorov called it the 'axiom of continuity'. He provided a number of axioms of continuity and proved their equivalence. These axioms were associated with the names of those prominent mathematicians of the 19th century due to whose work analysis acquired its coherence, viz: Dedekind, Bolzano, Weierstrass, Cantor, and Cauchy [36].

These axioms were as follows:

Section axiom

(i) Dedekind's axiom of section. If set R is presented as a union of two non-vacuous non-overlapping sets X and Y where each element of X is less than any element of Y, then there exists element z with a property that $x \leq z \leq y$ for any $x \in X$ and $y \in Y$ (or, equivalently, there is either a maximum element in X or a minimum element in Y).

Upper bound axiom

(ii) Bolzano's least upper (greatest lower) bound axiom. Any set $S \subset R$ bounded from above (below) has the least upper (greatest lower) bound (i.e. element M (m) with such property that $x \leq M, \forall x \in S$ and $x \geq m, \forall x \in S$, and for any $\varepsilon > 0$ there exists element $y \in S$ and $z \in S$) with such property that $y > M - \varepsilon$ and $z < m + \varepsilon$.

Limiting point axiom

(iii) Weierstrass's limiting point axiom. Any bounded sequence of elements from R has a limiting point (i.e. an element $\zeta \in R$, such that any ε-neighborhood of ζ contains an element of the sequence other than ζ).

Converging subsequence axiom

(iv) Weierstrass's converging subsequence axiom. A converging subsequence can be selected from any bounded sequence of elements from R.

Monotonic sequence axiom

(v) Bolzano's monotonically increasing or decreasing sequence. A bounded monotonically increasing or decreasing sequence of elements from R has a limit.

Nested intervals axiom

Cantor's axiom of nested intervals arises as a consequence: a sequence of nested intervals $\Delta_n = [a_n, b_n] \subset R$, $\Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \cdots$ with lengths tending to zero i.e. $b_n - a_n \to 0$ has a unique common point $\zeta \in \Delta_n, \forall n \in \mathbb{N}$ [36].

16 | Conclusion

In 1872, Dedekind articulated his axiom [37], while Bolzano's writings from the 1830s [23, 38, 39] were the first to mention the idea of a section. Bolzano proposed his postulate on the existence of the least upper bound in 1817 [29], but he was only able to demonstrate the coherence of the premise that the upper bound existed since there was no theory of real numbers. Cantor first proposed the concept of a limiting point in 1872 [40], and Weierstrass expanded on it in his lectures [41]. Based on Cantor's concept and discussions with Weierstrass, H.E. Heine initially proposed the idea of a convergent subsequence in 1872 [42]. We may observe the rise up the ladder of meanings from the ancient world to the present by looking at the history of the nested interval approach. Kepler and Cavalieri's interest in indivisibles was rekindled by the Renaissance, which also inspired Gregory to combine techniques and develop the concept of convergence. Calculus, or mathematics of variables, was created by Newton and Leibniz. MacLaurin was the first to highlight the importance of the Archimedes axiom. Additionally, he developed a metaphor—a sequence of nested intervals—that illustrates how sequences converge to one another. With the aid of the concepts of the upper bound, convergent sequences, and section, Bolzano's development of the idea of a function—and especially a continuous function—determined the lines of development of the concept of continuity. Mathematical analysis of the early 19 th century was based on Cauchy's theory of limits, sufficient at that time, and his theorem on continuous functions. Based thereon, a couple of concepts of a number and continuity appeared in the second half of the 19th century: those of Méray, Weierstrass, Cantor, and Dedekind. Hilbert found a solution to the differences in these concepts, having proved their equivalence; and in the 20 th century, Kolmogorov developed a unified concept of a real number.

17 |Declarations

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