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Quantitative Analysis of Co-current Imbibition Phenomenon with Heterogeneity and Gravitational Effects

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Abstract

The phenomenon of Co-current imbibition in a vertical download direction, coupled with the influence of heterogeneity and gravitational effects, is investigated through a comprehensive mathematical analysis in this research paper. Imbibition, the process of fluid penetration into porous media, plays a pivotal role in various natural and engineered processes. This study focuses on quantitatively understanding the behavior of Co-current imbibition and its interactions with heterogeneity and gravitational effects. The mathematical framework yields a non-linear partial differential equation (PDE) that describes the phenomenon. To solve this governing equation, the Variational Iteration Method is employed, along with appropriate initial and boundary conditions. The results are presented through numerical computations and graphical representations, facilitated by MATLAB.

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Introduction

Spontaneous imbibition, which is propelled by capillary force, in oil recovery process. This spontaneous (natural) imbibition whether in the format of Counter – current or Co – current imbibition, hold significant importance. The non-miscible and miscible phases flow in the similar direction in Co-current imbibition, with the non-miscible phase being driven out ahead of the miscible phase. In the case of Counter – current imbibition, the non-miscible and miscible phases travel in the reverse directions. For hydrophilic porous materials, imbibition is frequently thought to be Counter – current. Oil retrieval is propelled by Co-current imbibition, instead of Counter-current imbibition, in cases where a porous media is partially saturated with the wetting phase. Co-current imbibition is more effectual than Counter-current imbibition in the process of oil retrieval. Depending on the porous material and water injection rate, imbibition can take the form of either Co-current or Counter-current. If one end of the matrix is submerged in water and the another in oil, imbibition is Co-current, with water inflowing from one end and oil exiting from the other [15]. Imbibition is Counter-current from both ends due to an oil-covered porous matrix uncovered on both ends to water [15].
This phenomenon has recently attracted a lot of attention, particularly in the fields of petroleum engineering, reservoir engineering, geophysics, and hydrogeology. Several researchers investigated this phenomenon from numerous perspectives. Bourbiaux and Kalaydjian (1990) explored the experimental research of Counter-current as well as Co-current flow in natural porous media [4]. McWhorter and Sunada (1990) found integral

exact solution for the unstable horizontal movement of two incompressible, viscous liquids [6]. Pooladi-Darvish and Firoozabadi (2000) explored similarities and dissimilarities among Co-current and Counter-current imbibition, as well as their implications for practical applications [14]. Fazeli et. al. (2012) conferred Counter-current as well as Co-current imbibition in cracked porous medium exploiting the Homotopy perturbation technique (HPT) [7]. Yadav and Mehta (2014) used a small parameter method to derive an approximate analytical formula for Co-current imbibition during non-miscible double phase movement via porous media [15]. Patel and Desai (2017) proposed a mathematical model for the Co-current imbibition process in homogeneous inclined porous media, as well as its solution using the Homotopy analysis method (HAM) [12]. Jafari et. al. (2019) presented a solution analytically for one-dimensional horizontal imbibition in a Co-current flow problem [8]. Parikh and Shaikh (2020) constructed a mathematical model for Co-current imbibition in a homogeneous vertical porous media, which they solved using the Polynomial-based differential quadrature technique (PDQM) [2]. Shah and Parikh (2021) used the Variational iteration method (VIM) to examine the Counter-current imbibition process in vertical downward direction with heterogeneity influence [5].

This study is concerned with the phenomenon of Co-current imbibition, which ensues when incompatible fluids (oil and water) run through heterogeneous porous material while water is inserted vertically downwards with gravitational influence as well as mean capillary force.

The governing equation for this phenomenon is a one-dimensional non-linear PDE of second-order. The problem is handled by exploiting the method of Variational iteration [10, 11] with appropriate initial as well as boundary conditions. No one had studied this topic with heterogeneity and gravitational influence in the vertical downward direction.

The objective of this study is to reveal the saturation of wetting fluid water during the occurrence of Co – current imbibition in the oil retrieval process.

Problem Description

It is anticipated that during the oil retrieval process, water is infused into an oil-covered vertical heterogeneous porous medium. In the analyzed flow system, the velocity of oil and water is taken into account under the gravitational influence. We presumed that the permeability and porosity of the vertical heterogeneous porous medium are just functions of variable z for the sake of quantitative analysis. Scheidegger and Johnson [1] suggested the schematic representation (Fig. 1) of the finger. As a result, only the average cross-sectional region employed through fingers is chosen into consideration, with the shape and size of particular fingers being neglected. The saturation of wetting fluid water S_w at depth z and time t is thus expressed as the average cross-sectional region employed through wetting fluid water.

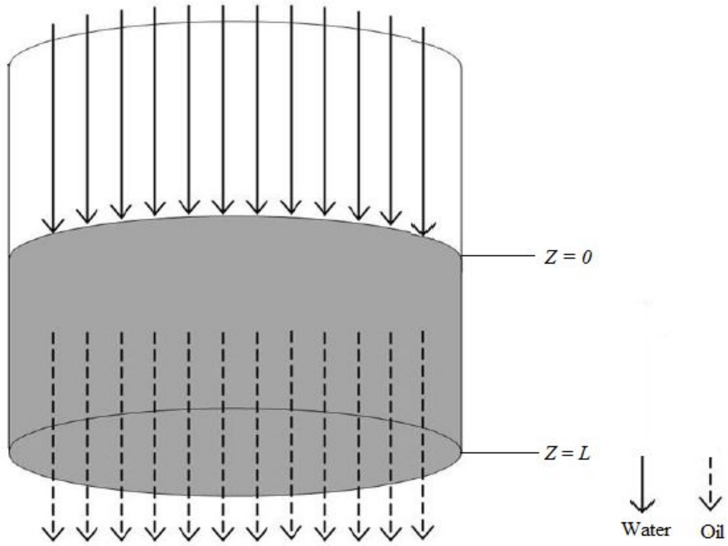

Fig. 1: Schematic view of Co-current imbibition [7]

Mathematical framework

The velocities of miscible fluid (water) and non – miscible fluid (oil) are expressed by the law of Darcy as [1], [9]:

$$V_w = -\frac{K_w}{\mu_w} K\left(\frac{\partial P_w}{\partial z} + \rho_w g\right) \tag{1}$$

$$V_o = -\frac{K_o}{\mu_o} K\left(\frac{\partial P_o}{\partial z} + \rho_o g\right) \tag{2}$$

For incompressible flow, the principle of conservation of mass confers

$$\phi \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial z} = 0 \tag{3}$$

The capillary pressure varies with phase saturation, which we accept as an empirical fact:

$$P_c\left(S_w\right) = P_o - P_w \tag{4}$$

The capillary pressure varies linearly based on the phase saturation [13].

$$P_c\left(S_w\right) = -\beta S_w \tag{5}$$

Assume the following analytical correlation among relative permeability and phase saturation [1].

$$K_w = S_w,\ K_o = 1 - \alpha S_w \tag{6}$$

In the Co-current imbibition phenomenon, total velocity $\left(V_t\right)$ equals the summation of the velocities of oil $\left(V_o\right)$ and water $\left(V_w\right)$.

$$V_o + V_w = V_t \tag{7}$$

We presume permeability and porosity as functions of variable z only for the analysed flow system in heterogeneous porous media [3].

Permeability $K\left(z\right) = K_o\left(1 + bz\right)$

Porosity $\phi(z)=\dfrac{1}{a_1-a_2z}$

For the purpose of simplicity, we take into account $K\propto\phi$ [16],

$$K=K_c\phi \tag{8}$$

Using Eq. (1), Eq. (2) and Eq. (7),

$$\frac{K_w}{\mu_w}K\left(\frac{\partial P_w}{\partial z}+\rho_w g\right)+\frac{K_o}{\mu_o}K\left(\frac{\partial P_o}{\partial z}+\rho_o g\right)=-V_t \tag{9}$$

Substituting Eq. (4) into Eq. (9),

$$\left(K\frac{K_w}{\mu_w}+K\frac{K_o}{\mu_o}\right)\frac{\partial P_w}{\partial z}+K\frac{K_o}{\mu_o}\frac{\partial P_c}{\partial z}+K\left(\rho_w\frac{K_w}{\mu_w}+\rho_o\frac{K_o}{\mu_o}\right)g=-V_t \tag{10}$$

Solving Eq. (10) for $\dfrac{\partial P_w}{\partial z}$

$$\frac{\partial P_w}{\partial z}=-\left(K\frac{K_w}{\mu_w}+K\frac{K_o}{\mu_o}\right)^{-1}\left[K\frac{K_o}{\mu_o}\frac{\partial P_c}{\partial z}+K\left(\rho_w\frac{K_w}{\mu_w}+\rho_o\frac{K_o}{\mu_o}\right)g+V_t\right] \tag{11}$$

Using Eq. (1) and Eq. (11), we get

$$V_w=-\frac{K_w}{\mu_w}\left(\frac{K_w}{\mu_w}+\frac{K_o}{\mu_o}\right)^{-1}\left[K\frac{K_o}{\mu_o}(\rho_w-\rho_o)g-K\frac{K_o}{\mu_o}\frac{\partial P_c}{\partial z}-V_t\right] \tag{12}$$

The water pressure can be given as,

$$P_w=\left(\frac{P_w+P_O}{2}\right)-\left(\frac{P_O-P_w}{2}\right)=\overline{P}-\frac{1}{2}P_c \tag{13}$$

Here the mean pressure ($\overline{P}$) is constant, so Eq. (10) indicates,

$$K\left(\rho_w\frac{K_w}{\mu_w}+\rho_o\frac{K_o}{\mu_o}\right)g+\frac{K}{2}\left(\frac{K_o}{\mu_o}-\frac{K_w}{\mu_w}\right)\frac{\partial P_c}{\partial z}=-V_t \tag{14}$$

Therefore Eq. (12) reduces to

$$V_w=\frac{K}{2}\frac{K_w}{\mu_w}\frac{\partial P_c}{\partial z}-K\frac{K_w}{\mu_w}\rho_w g \tag{15}$$

Using Eq. (!5) and Eq. (3), we get

$$\phi\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial z}\left(\frac{K}{2}\frac{K_w}{\mu_w}\frac{\partial P_c}{\partial z}-K\frac{K_w}{\mu_w}\rho_w g\right)=0$$

Using Eq. (15) and Eq. (3), we get

$$\phi\frac{\partial S_w}{\partial t}+\frac{\partial}{\partial z}\left[\frac{K}{2}\frac{K_w}{\mu_w}\frac{\partial P_c}{\partial z}-K\frac{K_w}{\mu_w}\rho_w g\right]=0 \tag{16}$$

Since $K=K_c\phi$, $K_w=S_w$ and $P_c=-\beta S_w$, we have

$$\frac{\partial S_w}{\partial t}=\frac{\beta}{2}\frac{K_c}{\mu_w}\left[\frac{\partial}{\partial z}\left(S_w\frac{\partial S_w}{\partial z}\right)+S_w\frac{\partial S_w}{\partial z}\frac{1}{\phi}\frac{\partial \phi}{\partial z}\right]+\frac{K_c\rho_w g}{\mu_w}\left(\frac{\partial S_w}{\partial z}+S_w\frac{1}{\phi}\frac{\partial \phi}{\partial z}\right) \tag{17}$$

Since

$$\frac{1}{\phi}\frac{\partial \phi}{\partial z}=\frac{\partial(\log\phi)}{\partial z}=\frac{\partial}{\partial z}\left(-\log a_1+\frac{a_2}{a_1}z\right)=\frac{a_2}{a_1}$$ (Omitting higher order terms of z)

By means of dimensionless variables,

$$Z=\frac{z}{L},\ T=\frac{\beta K_c t}{2\mu_w L^2}$$

Eq. (17) becomes

$$\frac{\partial S_w}{\partial T}=\frac{\partial}{\partial Z}\left(S_w\frac{\partial S_w}{\partial Z}\right)+A\frac{\partial S_w}{\partial z}+BS_w\frac{\partial S_w}{\partial Z}+ABS_w \tag{18}$$

Here $A=\dfrac{2L\rho_w g}{\beta}$, $B=\dfrac{a_2 L}{a_1}$, $S_w(z,\ t)=S_w(Z,\ T)$

Eq. (18) represents the governing non-linear PDE for the Co-current imbibition process in vertical downward direction with heterogeneity influence.

The proper initial and boundary conditions are listed below:

$S_w(Z,\ 0)=S_{w_0}(Z)$; for $Z>0$

$S_w(0,\ T)=S_{w_1}(T)$; for $T>0$

$S_w(L,\ T)=S_{w_2}(T)$; for $T>0$

Problem Solution

According to the approach of Variational iteration, the correction functional of Eq. (18) is written by,

$$S_{w_{n+1}}(Z,\ T)=S_{w_n}(Z,\ T)+\int_0^T\lambda(\tau)\left[\frac{\partial S_{w_n}(Z,\ \tau)}{\partial\tau}-\frac{\partial}{\partial Z}\left(S_{w_n}(Z,\ \tau)\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}\right)-A\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}\right.$$
$$\left.-BS_{w_n}(Z,\ \tau)\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}-ABS_{w_n}(Z,\ \tau)\right]d\tau$$

$S_{w_n}(Z,\ \tau)$ denotes restricted variation.

This means $\delta S_{w_n}(Z,\ \tau)=0$.

Obtaining variation with respect to S_{w_n}, remarking that $\delta S_{w_n}(0)=0$, produces

$$\delta S_{w_{n+1}}(Z,\ T)=\delta S_{w_n}(Z,\ T)+\delta\int_0^T\lambda(\tau)\left[\frac{\partial S_{w_n}(Z,\ \tau)}{\partial\tau}-\frac{\partial}{\partial Z}\left(S_{w_n}(Z,\ \tau)\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}\right)\right.$$
$$\left.-A\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}-BS_{w_n}(Z,\ \tau)\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}-ABS_{w_n}(Z,\ \tau)\right]d\tau$$

By means of integration by parts,

$$\delta S_{w_{n+1}}(Z,\ T)=\delta S_{w_n}(Z,\ T)+\left[\lambda(\tau)\delta S_{w_n}(Z,\ \tau)\right]_{\tau=T}-\int_0^T \lambda'(\tau)\delta S_{w_n}(Z,\ \tau)\ d\tau$$
$$-\int_0^T \lambda(\tau) ABS_{w_n}(Z,\ \tau)\ d\tau$$

Then, as shown below, it is possible to determine stationary conditions.

$\left[1+\lambda(\tau)\right]_{\tau=T}=0$,

$\left[-AB\lambda(\tau)-\lambda'(\tau)\right]_{\tau=T}=0$

So, $\lambda=-\exp\left[AB(T-\tau)\right]$

It is possible to find the following iteration formula.

$$S_{w_{n+1}}(Z,\ T)=S_{w_n}(Z,\ T)-\int_0^T \exp\left[AB(T-\tau)\right]\left[\frac{\partial S_{w_n}(Z,\ \tau)}{\partial \tau}-\frac{\partial}{\partial Z}\left(S_{w_n}(Z,\ \tau)\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}\right)\right.$$
$$\left.-A\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}-BS_{w_n}(Z,\ \tau)\frac{\partial S_{w_n}(Z,\ \tau)}{\partial Z}-ABS_{w_n}(Z,\ \tau)\right]d\tau \tag{19}$$

Putting $n=0$ in Eq. (19),

$$S_{w_1}(Z,\ T)=S_{w_0}(Z,\ T)-\int_0^T \exp\left[AB(T-\tau)\right]\left[\frac{\partial S_{w_0}(Z,\ \tau)}{\partial \tau}-\frac{\partial}{\partial Z}\left(S_{w_0}(Z,\ \tau)\frac{\partial S_{w_0}(Z,\ \tau)}{\partial Z}\right)\right.$$
$$\left.-A\frac{\partial S_{w_0}(Z,\ \tau)}{\partial Z}-BS_{w_0}(Z,\ \tau)\frac{\partial S_{w_0}(Z,\ \tau)}{\partial Z}-ABS_{w_0}(Z,\ \tau)\right]d\tau$$

We take the initial approximation,

$$S_{w_0}=e^{-z} \tag{20}$$

We obtain the following approximation by inserting the above approximation Eq. (20) into an iterative formula.

$$S_{w_1}=e^{-z}+\frac{1}{AB}\left(2e^{-2z}-Be^{-2z}-Ae^{-z}+ABe^{-z}\right)\left(e^{ABT}-1\right)$$

$$
\begin{aligned}
S_{w_2} &= e^{-z} + \frac{1}{AB}\left(2e^{-2z} - Be^{-2z} - Ae^{-z} + ABe^{-z}\right)\left(e^{ABT} - 1\right) \\
&\quad + \frac{1}{(AB)^2}\Big[\left(18e^{-3z} - 9Be^{-3z} - 4Ae^{-2z} + 4ABe^{-2z}\right) \\
&\qquad - A\left(4e^{-2z} - 2Be^{-2z} - Ae^{-z} + ABe^{-z}\right) - B\left(6e^{-3z} - 3Be^{-3z} - 2Ae^{-2z} + 2ABe^{-2z}\right) \\
&\qquad + 2AB\left(2e^{-2z} - Be^{-2z} - Ae^{-z} + ABe^{-z}\right)\Big]\left(e^{ABT} - 1\right)^2 \\
&\quad + \frac{1}{(AB)^3}\Big[\left(2e^{-2z} - Be^{-2z} - Ae^{-z} + ABe^{-z}\right)\left(8e^{-2z} - 4Be^{-2z} - Ae^{-z} + ABe^{-z}\right) \\
&\qquad + \left(4e^{-2z} - 2Be^{-2z} - Ae^{-z} + ABe^{-z}\right)^2 \\
&\qquad - B\left(2e^{-2z} - Be^{-2z} - Ae^{-z} + ABe^{-z}\right)\left(4e^{-2z} - 2Be^{-2z} - Ae^{-z} + ABe^{-z}\right)\Big]\left(e^{ABT} - 1\right)^3
\end{aligned} \tag{21}
$$

Similarly, the iterative formula Eq. (19) can be used to acquire more iterations.

Eq. (21) is the analytical solution of the governing non-linear PDE Eq. (18) expressing the saturation of wetting fluid water in a vertical downward direction with heterogeneity effect during the Co-current phenomenon.

Graphical and Numerical presentation

From the standard literature, the following constants are yielded:

Gravity $(g) = 9.8$, Density of water $(\rho_w) = 0.1$, Length $(L) = 1$, $\beta = 2$

Thus, $A = \dfrac{2L\rho_w g}{\beta} \approx 1$.

MATLAB is used to calculate the numerical data and create the graphical depiction of the solution Eq. (21). Table 1 shows the numerical data for the saturation of wetting fluid water S_w at different depth Z for specific time $T = 0,\ 0.02,\ 0.04, \ldots\ldots, 0.1$. Fig. 2 indicates the plots for the saturation of wetting fluid water S_w at different depth Z for specific time $T = 0,\ 0.02,\ 0.04, \ldots\ldots, 0.1$. Fig. 3 represents the plots for the saturation of wetting fluid water S_w at different time T for specific depth $Z = 0.1,\ 0.2,\ 0.3, \ldots\ldots, 1.0$. Fig. 4 displays the 3$D$ plot for the saturation of wetting fluid water S_w versus time $T = 0,\ 0.02,\ 0.04, \ldots\ldots, 0.1$ and depth $Z = 0.1,\ 0.2,\ 0.3, \ldots\ldots, 1.0$.

$T \rightarrow$	0	0.02	0.04	0.06	0.08	0.1
$Z \downarrow$	S_w					
0.1	9.048×10^{-1}	9.214×10^{-1}	9.383×10^{-1}	9.555×10^{-1}	9.730×10^{-1}	9.909×10^{-1}
0.2	8.187×10^{-1}	8.323×10^{-1}	8.461×10^{-1}	8.602×10^{-1}	8.746×10^{-1}	8.892×10^{-1}
0.3	7.408×10^{-1}	7.519×10^{-1}	7.632×10^{-1}	7.748×10^{-1}	7.865×10^{-1}	7.985×10^{-1}
0.4	6.703×10^{-1}	6.794×10^{-1}	6.887×10^{-1}	6.981×10^{-1}	7.077×10^{-1}	7.176×10^{-1}
0.5	6.065×10^{-1}	6.140×10^{-1}	6.215×10^{-1}	6.293×10^{-1}	6.372×10^{-1}	6.452×10^{-1}
0.6	5.488×10^{-1}	5.549×10^{-1}	5.611×10^{-1}	5.674×10^{-1}	5.739×10^{-1}	5.805×10^{-1}
0.7	4.966×10^{-1}	5.016×10^{-1}	5.066×10^{-1}	5.118×10^{-1}	5.171×10^{-1}	5.225×10^{-1}
0.8	4.493×10^{-1}	4.534×10^{-1}	4.576×10^{-1}	4.618×10^{-1}	4.661×10^{-1}	4.706×10^{-1}
0.9	4.066×10^{-1}	4.099×10^{-1}	4.133×10^{-1}	4.168×10^{-1}	4.203×10^{-1}	4.240×10^{-1}
1.0	3.679×10^{-1}	3.706×10^{-1}	3.734×10^{-1}	3.762×10^{-1}	3.792×10^{-1}	3.821×10^{-1}

Table 1: Saturation of water S_w for Co-current imbibition

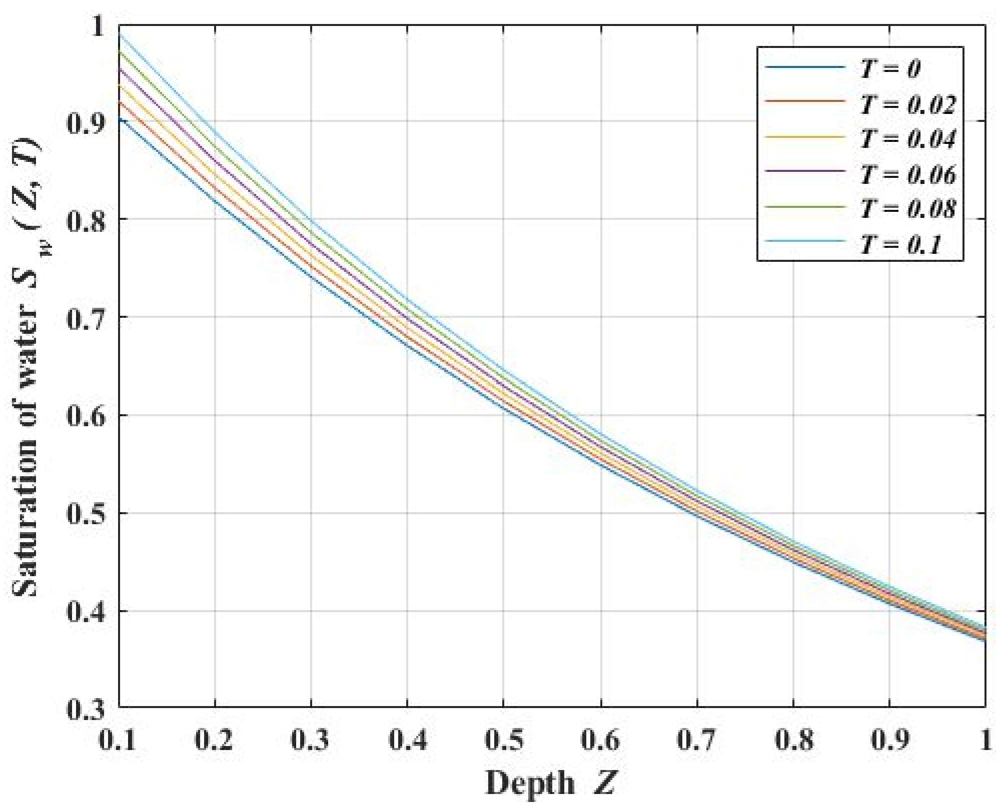

Fig. 2: 2D plot for the saturation of water Vs. Depth for fixed time

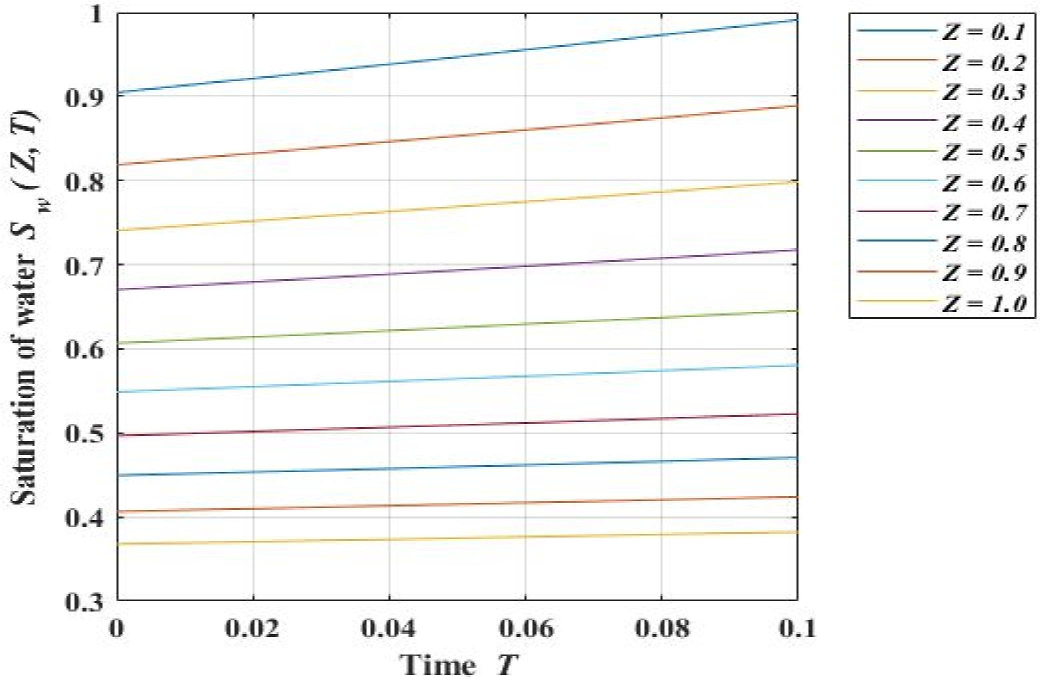

Fig. 3: 2D plot for the saturation of water Vs. Time for fixed depth

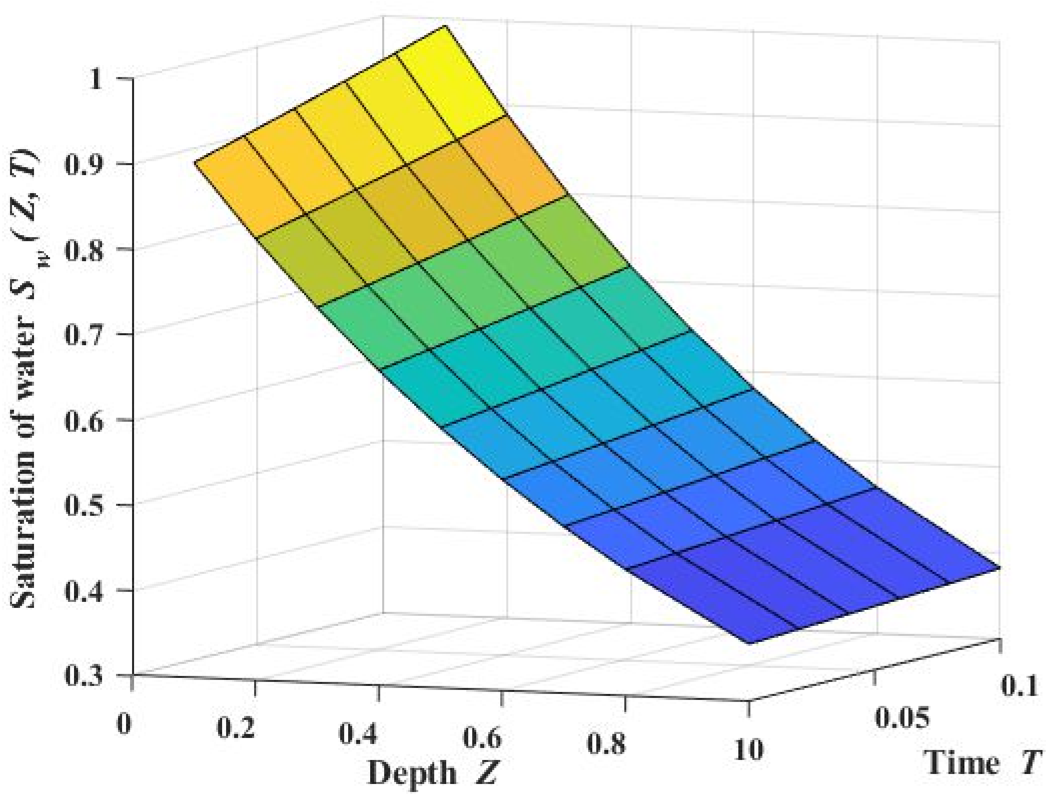

Fig. 4: 3D plot for the saturation of water Vs. Time and Depth

Conclusion

The Co-current imbibition phenomeon in a vertical heterogeneous porous medium is described mathematically in the current study. Eq. (18) represents the governing nonlinear PDE for this phenomeon. The solution Eq. (21) of Eq. (18) with appropriate boundary and initial conditions was found by applying the Variational iteration approach. Table 1 denotes numerical data for the saturation of water at different depth for different time. The saturation of wetting fluid water grows with time and diminishes with depth, as seen in Table 1. As a result, oil from the oil-formatted area will be pushed downward. Because the bottom is impermeable, the maximum amount of oil can be retained. It can enter into an oil producing well via connecting pipes during the oil recovery process. Fig. 2 exhibits the solution graphically with depth *Z*. Fig 3 indicates the solution graphically with time *T*. Fig. 4 displays a 3D graph of the solution. The saturation of wetting fluid water diminishes with depth as well as grows with time, as seen in Fig. 2, Fig. 3 and Fig. 4, which is consistent with the physical behaviour of the current study.

Nomenclatures

V_o : Velocity of oil	ϕ : Porosity
V_w : Velocity of water	S_o : Saturation of oil
K_o : Relative permeability of oil	S_w : Saturation of water
K_w : Relative permeability of water	K_c, β : Constant of proportionality
K : Permeability of heterogeneous porous medium	a_1, a_2, b, K_0 : Constants
ρ_o : Density of oil	L : Length of cylindrical porous matrix
ρ_w : Density of water	λ : Lagrange multiplier
P_o : Pressure of oil	S_n : Restricted variation
P_w : Pressure of water	T : Time
μ_o : Kinematic viscosity of oil	Z : Depth
μ_w : Kinematic viscosity of water	

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