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Regular Long Wave Equation Solution of Fractional Black-Scholes Option Pricing Model

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Abstract
The traditional Black-Scholes equation, guided by Brownian motion, lacks memory. Therefore, it is deemed appropriate to substitute Brownian motion with fractional Brownian motion (FBM), characterized by long-memory dynamics attributed to the Hurst exponent. This paper focuses on deriving the option pricing equation modeled with fractional Brownian motion. The equation is then transformed into a one-dimensional heat equation through the Shehu transform, and a solution is subsequently obtained. The Black-Scholes Model is a widely utilized tool for option pricing, a critical application in finance. In scenarios without transaction costs, option value is determined using the Black-Scholes model. In the context of the Caputo sense, this study proposes a solution for the fractional Black-Scholes equation (FBSE) problem. The research revisits the direct algebraic method initially proposed by Hereman et al. (1985) and applies this methodology to solve both the Benjamin-Bona-Mahony (RLW) equation (PDE) into an ordinary differential equation (ODE). Subsequently, the ODE is solved using algebraic processes, resulting in solutions for the Benjamin-Bona-Mahony equations and hence the solution of thefractional Black-Scholes option pricing model. Also the modified sine-cosine method is used to solve the non-homogeneous form of Benjamin-Bona-Mahony equation with time-dependent coefficients wasto solve non-homogeneous case of our problem.

Keyword: Fractional Black-Scholes Equation, Shehu Transform, Homotopy Analysis Shehu Transform Method, Homotopy Analysis Method, option pricing.

1. Introduction

Across various scientific disciplines, there has been a growing fascination with exploring systems that encompass memory or delayed effects, where the impact of delay on state equations is a notable consideration. In numerous mathematical models, phenomena often exhibit long-range memories, potentially influenced by factors such as extreme weather or natural disasters. In certain instances, stochastic dynamical systems not only depend on present and past states but also involve derivatives with delays. In these scenarios, a category of stochastic differential equations driven by fractional Brownian motion emerges as a crucial tool for effectively describing and analysing such complex systems.

Fractional Brownian motion of Hurst exponent $H \in (0, 1)$ is a stochastic process $\{B_H(t), t \in \mathbb{R}\}$ which satisfies the following:

1. $B_H(t)$ is Gaussian, that is, for every t > 0, $B_H(t)$ has a normal distribution
2. $B_H(t)$ is a self similar process meaning that for any $\propto > 0$, $B_H(\propto t)$ has the same law as $\propto^H B_H(t)$.
3. It has stationary increments, that is, $B_H(t)$ - $B_H(s) \sim B_H(t-s)$.

Fractional Brownian motion, initially introduced by Kolmogrov in 1940 and subsequently explored by Mandlbrov and Van Ness, exhibits various properties that have garnered attention. Its applications extend into the pricing of financial derivatives, where a derivative is an instrument whose valuation is dependent on or derived from the value of another asset, often a stock.The concept of financial derivatives is not a recent development. Though there arehistorical debates regarding the precise creation date of financial derivatives, it is widely acknowledged that Charles Castelly's work in 1877 marked the first attempt at modern derivative pricing. In 1969, Fisher Black and Myron Scholes introduced an idea that would revolutionize the world of finance. Their groundbreaking paper centered around the revelation that estimating the expected return of a stock was not necessary to price an option written on that stock

The Black-Scholes option pricing equation, typically driven by standard Brownian motion, is adapted in this study by replacing the conventional Brownian motion with fractional Brownian motion, characterized by the inclusion of the Hurst exponent denoted as H. The Hurst exponent is a statistical measure employed for time series classification, with its values ranging between 0 and 1. This modification allows for a more nuanced modeling of financial markets and enhances the understanding of option pricing dynamics.

Some of the authors have conducted studies on option pricing dynamicsinclude Ouafoudi and Gao (2018). The authors employed the Modified Homotopy Perturbation Method (MHPM), Homotopy Perturbation Method (HPM), and Sumudu transforms to address the fractional Black-Scholes (B-S) equation. The results obtained from both approaches were found to be consistent.

Yavuz and Ozdemir (2018) tackled the fractional B-S equation by recalibrating it as a fractional mean and applying the Adomian Decomposition Method (ADM) to both the fractional and generalized B-S equations to calculate option prices for fractional values.

Alfaqeih and Ozis (2020) utilized an Aboodh transform and the Adomian Decomposition Method to solve a fractional B-S equation .

Fadugba and Edogbanya (2020) conducted a comparative analysis of the fractional Laplace transform homotopy perturbation method and the fractional reduced differential transform method for solving the time-fractional B-S equation. The study found that the Fractional Reduced Differential Transform Method (FRDTM) outperformed the Fractional Laplace Transform Homotopy Perturbation Method (FLTHPM) due to its shorter algorithms.

To obtain an analytical solution for the time-space fractional B-S method, Edeki et al. (2020) employed a coupled transform approach, combining the characteristics of the fractional complex transform and reduced differential transform methods.

Bhadane et al. (2020) presented analytical solutions for the fractional B-S equation using a combination of the Homotopy Perturbation Method (HPM) and the Elzaki transform, known as the Elzaki transform HPM.

In addressing the fractional B-S problem, Ahmad et al. (2021) introduced a modified version of the Differential Transform Method (DTM) called the Fractional Reduced Differential Transform Method.

Yavuz and Ozdemir (2018) employed an iterative method to derive an approximate solution for fractional Black-Scholes models in a conformable derivative sense.

Shehu and Zhao (2019) applied a new semi-analytical method known as the Homotopy Analysis Shehu Transform Method (HASTM) to solve the multidimensional fractional diffusion equation. This method not

only reduces the need for iterative differentiation and integration but also overcomes the restrictions of the Homotopy Analysis Method (HAM). Consequently, this inspired the application of the Homotopy Analysis Shehu Transform to the given B-S problem. The solutions are extremely consistent withthe existing results.
In this paper we intend to solve the Black-Scholes option pricing equation modeled by fractional Brownian motion using direct algebraic method of solution of the Benjamin–Bona–Mahony Regular Long Wave (RLW) Equation.,

2. Fractional Option Pricing Model

Theorem 1: Let a generic payoff function $G(t) = U(S, t)$. Then the partial differential equation associated with the price of the derivative on the stock price is

$$\frac{\partial U}{\partial t} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} - rU = 0, \quad S > 0,\ t > 0\ H \in (0,1),\ H \neq \frac{1}{2}, \quad (1)$$

where U is the call option price , t is the time to maturity, H is the Hurst exponent, σ is the volatility , S is the stock price and r is the discount rate.

Proof:

The stock price S_t follows the fractional Brownian motion process

$$dS = \mu S dt + \sigma S dB_H(t). \quad (2)$$

The wealth of an investor Vfollows a diffusion process given by

$$dU = \Lambda dS + r(X - \Lambda S)dt. \quad (3)$$

Putting equation (2) into equation (3) yields

$$dU = \{rX + \Lambda S(\mu - r)\}dt + \Lambda S\sigma dB_H(t), \quad (4)$$

where μ –r is the risk premium.

Suppose that the value of this claim at time t is given by

$G(t) = U(S,t), \quad S = S_t$(5)

Applying the Ito's formula for fractional Brownian motion on equation (5), we have

$$dG = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}dS + Ht^{2H-1}\frac{\partial^2 U}{\partial S^2}(dS)^2 \ (6)$$

Substituting (2) in (6), we have

$$dG = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}[\mu S dt + \sigma S dB_H(t)] + Ht^{2H-1}\frac{\partial^2 U}{\partial S^2}[\mu S dt + \sigma S dB_H(t)]^2. (7)$$

Equation (7) simplifies to

$$dG = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}[\mu S dt + \sigma S dB_H(t)]$$

$$+Ht^{2H-1}\frac{\partial^2 U}{\partial S^2}[\mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 dt dB_H(t) + \sigma^2 S^2 (dB_H(t))^2], \ (8)$$

Using the multiplication rule that

$dt dB_H(t) = (dt)^2 = 0;\ (dB_H(t))^2 = dt$, (Bernard,2007; Chukwueze et al 2019).

Therefore (8) reduces to

$$dG = \frac{\partial U}{\partial t}dt + \frac{\partial U}{\partial S}[\mu S dt + \sigma S dB_H(t)] + Ht^{2H-1}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}dt (9)$$

Collecting like terms we have

$$dG = \left[\frac{\partial U}{\partial t} + \mu S\frac{\partial U}{\partial S} + Ht^{2H-1}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}\right]dt + \sigma S\frac{\partial U}{\partial S}\, dB_H(t) \quad (10)$$

Using (5) we have

$$dU = \left[\frac{\partial U}{\partial t} + \mu S\frac{\partial U}{\partial S} + Ht^{2H-1}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}\right]dt + \sigma S\frac{\partial U}{\partial S}\, dB_H(t) \quad (11)$$

Thus, equating coefficients in (3) and (11), we have

$\frac{\partial U}{\partial t} + \mu S \frac{\partial U}{\partial S}\ H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 U}{\partial S^2} = rU + \Lambda_t S(\mu - r)$,(12)

from which

$$\sigma S \frac{\partial U}{\partial S} = \Lambda_t \sigma S \quad (13)$$

and

$\Lambda_t = \frac{\partial U}{\partial S}$.(14)

The substitution equation (14) into (12), gives

$\frac{\partial U}{\partial t} + \mu S \frac{\partial U}{\partial S}\ H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 U}{\partial S^2} = rU + S\mu \frac{\partial U}{\partial S} - Sr \frac{\partial U}{\partial S}$.(15)

This implies that

$$\frac{\partial U}{\partial t} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 U}{\partial S^2} + Sr \frac{\partial U}{\partial S} - rU = 0 \quad (16)$$

V(S,t) is the European call option price, S is the stock price at time t, t is the time to the expiration of the option, r is the discount rate, σ represents the volatility function of the underlying asset and H is the Hurst exponent.

3. The Model

Theorem 2: *Let equation (1) be given by*

$$\frac{\partial U}{\partial t} + Ht^{2H-1} S^2 \sigma^2 \frac{\partial^2 U}{\partial S^2} + rS \frac{\partial U}{\partial S} - rU = 0,\ S > 0, t > 0 \quad (17)$$

with $U(0,t) = 0$, $U(S,t) \sim S$ *as* $S \to \infty$, $U(S,T) = max\{|S - K|, 0\}$

Then (17) can be reduced to one-dimensional heat equation of the form

$\frac{\partial U}{\partial \tau} = p \frac{\partial^2 U}{\partial x^2}$. (18)

Set $\tau = \frac{\sigma^2 (T-t)}{2}$; $x = \ln(S/K)$ and

$U(S,t) = Kv(x,\tau)$. (19)

Differentiating (19), we have

$\frac{\partial U}{\partial t} = K \frac{\partial U}{\partial \tau} . \frac{\partial \tau}{\partial t} = \left(K \frac{\partial U}{\partial \tau}\right)\left(-\frac{\sigma^2}{2}\right)$ (20)

$\frac{\partial U}{\partial S} = K \frac{\partial U}{\partial x} . \frac{\partial x}{\partial S} = K \frac{\partial U}{\partial x}\left(\frac{1}{S}\right) = \frac{K}{S}\frac{\partial U}{\partial x}$. (21)

The second partial derivative of $U(S,t)$ with respect to S is given as

$$\frac{\partial^2 U}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial U}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial U}{\partial x}\right) = \frac{K}{S}\left(\frac{\partial}{\partial S}\frac{\partial U}{\partial x}\right) + \frac{\partial U}{\partial x}\left(\frac{\partial}{\partial S}\frac{K}{S}\right)$$

$= \frac{K}{S}\left(\frac{\partial}{\partial S}\frac{\partial U}{\partial x}\right) + \frac{\partial U}{\partial x}\left(\frac{-K}{S^2}\right)$

$= \frac{K}{S}\left[\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right)\frac{dx}{dS}\right] - \frac{K}{S^2}\frac{\partial U}{\partial x}$

$= \frac{K}{S}\frac{\partial^2 U}{\partial x^2}\left(\frac{1}{S}\right) - \frac{K}{S^2}\frac{\partial U}{\partial x}$,

from which we obtain

$\frac{\partial^2 U}{\partial S^2} = -\frac{K}{S^2}\frac{\partial U}{\partial x} + \frac{K}{S^2}\frac{\partial^2 U}{\partial x^2}$.(22)

The terminal condition is

$U(S,T) = \max\{|S - K|, 0\}$

$= \max\{|Ke^x - K|, 0\}$.

Let

$U(S,T) = Kv(x,0)$,

$$v(S,T) = \max\{|e^x - 1|, 0\}.(23)$$

Substitute (20), (21) and (22) in (1)

$$\left(K\frac{\partial U}{\partial \tau}\right)\left(\frac{\sigma^2}{2}\right) + H\left(T - \frac{2\tau}{\sigma^z}\right)^{2H-1} S^2\sigma^2\left(-\frac{K}{S^2}\frac{\partial U}{\partial x} + \frac{K}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + \mathrm{rS}\left(\frac{K}{S}\frac{\partial U}{\partial x}\right) - rKU = 0\,. \quad (24)$$

Let

$m = \frac{2\tau}{\sigma^2}$,

then

$$-\frac{\sigma^2}{2}\frac{\partial U}{\partial \tau} + H(T-m)^{2H-1}S^2\sigma^2\left(-\frac{1}{S^2}\frac{\partial U}{\partial x} + \frac{1}{S^2}\frac{\partial^2 U}{\partial x^2}\right) + rS\left(\frac{1}{S}\frac{\partial U}{\partial x}\right) - rU = 0. \quad (25)$$

$$-\frac{\sigma^2}{2}\frac{\partial \mathrm{U}}{\partial \tau} + H(T-m)^{2H-1}\sigma^2\frac{\partial U}{\partial x} + H(T-m)^{2H-1}\sigma^2\frac{\partial^2 U}{\partial x^2} + r\frac{\partial v}{\partial x} - rU = 0 \quad (26)$$

$$-\frac{\sigma^2}{2}\frac{\partial U}{\partial \tau} + H(T-m)^{2H-1}\sigma^2\frac{\partial U}{\partial x} - r\frac{\partial v}{\partial x} - H(T-m)^{2H-1}\sigma^2\frac{\partial^2 U}{\partial x^2} + rU = 0 \quad (27)$$

$$\frac{\sigma^2}{2}\frac{\partial U}{\partial \tau} + [H(T-m)^{2H-1}\sigma^2 - r]\frac{\partial U}{\partial x} - H(T-m)^{2H-1}\sigma^2\frac{\partial^2 U}{\partial x^2} + rU = 0 \quad (28)$$

$$\frac{\partial U}{\partial \tau} + \left[2H(T-m)^{2H-1} - \frac{2r}{\sigma^2}\right]\frac{\partial U}{\partial x} - 2H(T-m)^{2H-1}\frac{\partial^2 U}{\partial x^2} + \frac{2rU}{\sigma^2} = 0. \quad (29)$$

Let

$p = 2H(T-m)^{2H-1} = 1$ and $q = \frac{2r}{\sigma^2} = A$,

then we have

$$\frac{\partial U}{\partial \tau} - p\frac{\partial^2 U}{\partial x^2} + +(p-q)\frac{\partial U}{\partial x} + qU = 0,(30)$$

which is a homogenous partial differential equation of second order in x.

We shall solve equation (30) using direct algebraic method ofBenjamin-Bona–Mahony Regular Long Wave (RLW) Equation.

The Regularized Long-Wave (RLW) equation, originally proposed by Peregrine to depict undular bore development, gained significance as an enhancement of the Korteweg-de Vries equation (KdV equation) by Benjamin, Bona, and Mahony in 1972. It serves as a valuable model for long surface gravity waves of small amplitude propagating in unidirectional $(1+1)$ dimensions.

The solution process involves the following steps:

(i) Initiation with the non-linear equation in $1+1$dimensions, where x and t represent space and time coordinates, respectively. The introduction of a traveling frame of reference through $\varepsilon = x - vt$ transforms the given non-linear partial differential equation (PDE) in $U(x,t)$ into an ordinary differential equation (ODE) in $\phi(\varepsilon) \triangleq U(x,t)$.

(ii) Integration of the ODE with respect to ε as many times as possible, avoiding integral equations.

(iii) Substitution of $\phi = c_1 + \phi$ to obtain the most general solitary solution, potentially containing a constant term, c_1 .

(iv) Consideration of the linear part of the equation in ϕ by setting the coefficients of the nonlinear n term(s) equal to zero or neglecting them. By setting $\phi = e^{k\varepsilon}$ in the linear equation, values of k and the constant term c_1 are obtained by setting the ϕ̂-independent part equal to zero. These values are then substituted back into the equation formed in step

(v) Normalization of a few coefficients of the nonlinear terms for mathematical convenience by a single scaling transformation of ϕ into ϕ.

(vi) Solution of the nonlinear equation in ϕ by expanding ϕ in terms of the harmonics of the decaying exponential solution of the linear equation. Setting $g(x) = e^{-k\varepsilon}$ and $\tilde{\phi} = \sum_{n=1}^{\infty}\sum a_n g^n(\varepsilon)$, and applying Cauchy's rule for products, a recursion relation fora_ns is obtained.

(vii) Solution of the recursion relation to determine the general form of the coefficients a_n through direct algebraic processes and computations.
(viii) Substitution of the coefficients $a_n$obtained from the recursion relation into $\hat{\phi}$ and using the scaling factor $\phi = c_1 + \hat{\phi}$ to obtain the solution.
(ix) Return to the original dependent variable u and the independent variables x and t. An exact solitary wave solution of the nonlinear PDE is obtained.
Further insight and demonstration of the application of these steps are provided, focusing on the application of the Regular Long Wave (RLW) equation.

The application of the Benjamin–Bona–Mahony Regular Long Wave (RLW) Equation

The Benjamin–Bona–Mahony Regular Long Wave(RLW) equation is given below:
$U_t + U_x + \alpha U U_x - U_{2xt} = 0.$
We present a solution using the approach/methodology as follows,

$$U_t + U_x + \alpha U U_x - U_{2xt} = 0. \qquad (31)$$

To solve equation (31), we compare equations (30) and (31) and transform the resultant equationinto an ODE using the traveling frame of reference
$\varepsilon = x - vt, \quad \phi(\varepsilon) \triangleq u(x,t).$
$u_t = -v\phi_\varepsilon, u_x = \phi_\varepsilon, u_{2xt} = (u_{2x})_t = -v\phi_{3\varepsilon}.$ (32)
Assuming that $u_{2xt} = pu_{xx}$, substituting (32) into (30), we obtain
$-v\phi_\varepsilon + (p-q)\phi_\varepsilon + \alpha\phi\phi_\varepsilon + v\phi_{3\varepsilon} = 0\ .$ (33)
Integrating (33) with respect to ϵ we have,
$-v\int \phi_e d\varepsilon + (p-q)\int \phi_\varepsilon d\varepsilon + \alpha \int \phi\phi_\varepsilon d\varepsilon + v\int \phi_{3\varepsilon} d\varepsilon = 0,$
which with $\lambda = (p-q)$simplifies to
$-v\phi + \lambda\phi + \frac{\alpha}{2}\phi^2 + v\phi_{2\varepsilon} + c_1 C = 0$ (34)
where $c_1 C$ is the constant of integration.
Substituting

$$\phi = c_1 + \hat{\phi} \qquad (35)$$

into (34) we get,
$-v(c_1 + \hat{\phi}) + \lambda(c_1 + \hat{\phi}) + \frac{\alpha}{2}(c_1 + \hat{\phi})^2 + v(c_1 + \hat{\phi})_{2\varepsilon} + c_1 C = 0,$
that modifies to
$(-v + \lambda + \alpha c_1)\hat{\phi} + v\hat{\phi}_{2\varepsilon} + \frac{\alpha}{2}\phi^2 - vc_1 + c_1 + \frac{\alpha}{2}{c_1}^2 + c_1 C = 0.$ (36)
Ignoring the nonlinear part and setting the$\hat{\phi}$ −independent part equal to zero (0) the remaining linear part of equation (36) becomes,
$(-v + \lambda + \alpha c_1)\hat{\phi} + v\hat{\phi}_{2\varepsilon} = 0.$
Let, $\hat{\phi} = e^{k\varepsilon}$, and $\hat{\phi}_{2\varepsilon} = k^2 e^{k\varepsilon}$ then
$(-v + \lambda + \alpha c_1)e^{k\varepsilon} + vk^2 + vk^2 e^{k\varepsilon} = 0,$
from which we obtain

$$-v + \lambda + \alpha c_1 + vk^2 = 0$$

and

$$k^2 = \frac{v - \lambda - \alpha c_1}{v}.$$

Considering the $\hat{\phi}$ − independent part that has been set equal to zero, we get
$-vc_1 + c_1 + \frac{\alpha}{2}c_1^2 + c_1 C = 0,$

and

$$-v+\lambda+\frac{\alpha}{2}c_1+C=0$$

from which we obtain

$$c_1=\frac{2(v-\lambda-C)}{\alpha}, \qquad (37)$$

and

$$k^2=\frac{1}{v}\left[v-\lambda-\alpha\left(\frac{2(v-\lambda-C)}{\alpha}\right)\right]$$
$$=\frac{1}{v}[v-\lambda-2v+2\lambda+2C]$$

$$k=\sqrt{\frac{\lambda-v+2C}{pv}}. \qquad (38)$$

Substituting (37) in (66), we get

$$\left(-v+\lambda+\alpha\left[\frac{2(v-\lambda-C)}{\alpha}\right]\right)\hat{\phi}+v\hat{\phi}_{2\varepsilon}+\frac{\alpha}{2}\hat{\phi}^2=0.$$

and

$$(v-\lambda-2C)\hat{\phi}+v\hat{\phi}_{2\varepsilon}+\frac{\alpha}{2}\hat{\phi}^2=0. \qquad (39)$$

Normalizing the coefficient of the non linear term using the scaling term

$$\hat{\phi}=\frac{2}{\alpha}(\lambda+2C-v)\hat{\phi}. \qquad (40)$$

Then putting (40) in (39) yields

$$\frac{2}{\alpha}(v-\lambda-2C)(\lambda+2C-v)\tilde{\phi}+v\left[\frac{2}{\alpha}(\lambda+2C-v)\tilde{\phi}\right]_{2\varepsilon}+\frac{\alpha}{2}\left[\frac{2}{\alpha}(\lambda+2C-v)\tilde{\phi}\right]^2=0,$$

that simplifies to

$$(\lambda+2C-v)\tilde{\phi}-v\tilde{\phi}_{2\varepsilon}-(\lambda-2C-v)\tilde{\phi}^2=0. \qquad (41)$$

From (38)

$$vk^2=1+2C-v$$

and substituting this into (41) leads to

$$vk^2\tilde{\phi}-v\tilde{\phi}_{2\varepsilon}-vk^2\tilde{\phi}^2=0$$

that reduces to

$$k^2\tilde{\phi}-\tilde{\phi}_{2\varepsilon}-k^2\tilde{\phi}^2=0. \qquad (42)$$

The expansion of$\tilde{\phi}$ in terms of the harmonics of the decaying exponential solution of the linear equation $g(x)=e^{-k\varepsilon}$,

we have using theCauchy's rule that

$$\tilde{\phi}=\sum_{n=1}^{\infty}a_n g^n(\varepsilon)=\sum_{n=1}^{\infty}a_n e^{-nk(\varepsilon)}$$
$$\tilde{\phi}_{\varepsilon}=-k\sum_{n=1}^{\infty}na_n g^n(\varepsilon)$$
$$\tilde{\phi}_{2\varepsilon}=k^2\sum_{n=1}^{\infty}n^2a_n g^n(\varepsilon)$$
$$\tilde{\phi}^2=\left[\sum_{n=1}^{\infty}a_n g^n(\varepsilon)\right]^2$$
$$=\sum_{n=1}^{\infty}\sum_{l=1}^{n-1}a_l a_{n-l}g^n(\varepsilon).$$

Substituting these expansions into (42), it becomes

$$k^2\sum_{n=1}^{\infty}a_ng^n(\varepsilon)-k^2\sum_{n=1}^{\infty}n^2\,a_ng^n(\varepsilon)-k^2\sum_{n=1}^{\infty}\sum_{l=1}^{n-1}a_la_{n-l}g^n(\varepsilon)=0$$

$$\left[(1-n^2)a_n-\sum_{l=1}^{n-1}a_la_{n-l}\right]k^2\sum_{n=1}^{\infty}g^n(\varepsilon)=0$$

$(n^2-1)a_n+\sum_{l=1}^{n-1}a_la_{n-l}=0, n\geq 2.$ (43)

The recursion relation (43) is solved to find the general form of the coefficient a_n

For $n=2$,

$$(2^2-1)a_2+\sum_{l=1}^{1}a_la_{2-l}=0$$

$$3a_2+a_1a_1=0$$

$$a_2=\frac{1}{3}a_1^2=-\frac{6^2}{3}\left(\frac{a_1}{6}\right)^2$$

For $n=3$

$$(3^2-1)a_3+\sum_{l=1}^{2}a_la_{3-l}=0$$

$$8a_3+a_1a_2+a_2a_1=0$$

$$a_3=-\frac{1}{4}a_1\left(-\frac{1}{3}a_1^2\right)=\frac{1}{12}a_1^3=6(3)(-1)^{3+1}\left(\frac{a_1}{6}\right)^3$$

For $n=4$

$$a_4=-24\left(\frac{a_1}{6}\right)^4$$

$$=6(4)(-1)^{4+1}\left(\frac{a_1}{6}\right)^4$$

From the above, a_n is arbitrary and the pattern can be clearly seen as

$$a_n=6n(-1)^{n+1}\left(\frac{a_1}{6}\right)^n, a_1>0$$

Then the coefficient$a_n$is substituted into the equation for ϕ to obtain the solution.

From (4.5),

$$\phi=c_1+\hat{\phi}$$

But,

$\phi=\frac{2}{\alpha}(v-\lambda-C)+\hat{\phi}.$

Also, from (40),

therefore

$\phi=\frac{2}{\alpha}(v-1-C)+\frac{2}{\alpha}(1+2C-v)\tilde{\phi}.$

But,

$$\tilde{\phi}=\sum_{n=1}^{\infty}a_ng^n(\varepsilon)$$

$$\phi=\frac{2}{\alpha}(v-\lambda-C)+\frac{2}{\alpha}(\lambda+2C-v)\sum_{n=1}^{\infty}a_ng^n(\varepsilon)$$

$=\frac{2}{\alpha}(v-\lambda-C)+\frac{2}{\alpha}(\lambda+2C-v)\sum_{n=1}^{\infty}6n(-1)^{n+1}\left(\frac{a_1}{6}\right)^n g^n.$

Let$a=\frac{a_1}{6}$

therefore

$\phi=\frac{2}{\alpha}(v-\lambda-C)+\frac{2}{\alpha}(\lambda+2C-v)\sum_{n=1}^{\infty}6n(-1)^{n+1}(ag)^n.$ (44)

The power series $\sum_{n=1}^{\infty}6n(-1)^{n+1}(ag)^n$ is convergent for $ag<1$.

Using

Differentiating both sides the well-known power series

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n,|x|<1$$

with respect to $x$we have

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)$$

and

$$\frac{d}{dx}(1-x)^{-1}=\frac{d}{dx}\sum_{n=0}^{\infty}x^n$$

which becomes

$\frac{1}{(1-x)^2}=\sum_{n=1}^{\infty}nx^{n-1}.$

Further, we multiply both sides of the above equation by x, we get

$\frac{x}{(1-x)^2}=\sum_{n=1}^{\infty}nx^n.$

The substitution of $(-ag)$ for x, gives

$$-\frac{ag}{(1+ag)^2}=\sum_{n=0}^{\infty}n(-ag)^n$$

which modifies to

$-\frac{ag}{(1+ag)^2}=\sum_{n=0}^{\infty}n(-1)^{n+1}(ag)^n.$ (45)

Substituting (45) into (44), we have

$\phi=\frac{2}{\alpha}(v-\lambda-C)+\frac{2}{\alpha}(\lambda+2C-v)\frac{6ag}{(1+ag)^2}.$

Putting $g(x)=e^{-k\varepsilon}$ as defined initially, into the above equation, we get

$\phi=\frac{2}{\alpha}(v-\lambda-C)+\frac{2}{\alpha}(\lambda+2C-v)\frac{6ae^{-k\varepsilon}}{\left(1+ae^{-k\varepsilon}\right)^2}.$ (46)

We obtain equation from (46)

$$\phi=\frac{2}{\alpha}(v-\lambda-C)+\frac{12}{\alpha}(\lambda+2C-v)\left[\frac{a}{e^{k\varepsilon}}\div\left(1+\frac{a}{e^{k\varepsilon}}\right)^2\right]$$

$$=\frac{2}{\alpha}(v-\lambda-C)+\frac{12}{\alpha}(\lambda+2C-v)\times\frac{ae^{k\varepsilon}}{a^2\left(1+\frac{1}{a}e^{k\varepsilon}\right)^2}$$

$$=\frac{2}{\alpha}(v-\lambda-C)+\frac{12}{\alpha}(\lambda+2C-v)\times\frac{\frac{1}{a}e^{k\varepsilon}}{\left(1+\frac{1}{a}e^{k\varepsilon}\right)^2}$$

$=\frac{2}{\alpha}(v-\lambda-C)+\frac{12}{\alpha}(\lambda+2C-v)\times\frac{e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}}{\left(1+e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}\right)^2}.$ (47)

Also since

$$sechx=\frac{2e^x}{1+e^{2x}}$$

Hence,

$$\frac{1}{4}sech^2x = \frac{e^{2x}}{(1+e^{2x})^2}$$

Let

$$x = \frac{1}{2}\left(\ln\left(\frac{1}{a}\right) + k\varepsilon\right)$$

Then $\quad \frac{1}{4}sech^2x\left[\ln\left(\frac{1}{a}\right) + k\varepsilon\right] = \frac{e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}}{\left(1+e^{\ln\left(\frac{1}{a}\right)+k\varepsilon}\right)^2}$

Substituting this into (4.17), the solution $u(x,t)$ is obtained

$$\phi = \frac{2}{\alpha}(v-\lambda-C) + \frac{3}{\alpha}(\lambda+2C-v)sech^2\left[\frac{1}{2}\ln\left(\frac{1}{a}\right) + \frac{1}{2}k\varepsilon\right]$$

Let

$$\vartheta = \frac{1}{2}\ln\left(\frac{1}{a}\right)$$

$$\phi = \frac{2}{\alpha}(v-\lambda-C) + \frac{3}{\alpha}(\lambda+2C-v)sech^2\left[\vartheta + \frac{1}{2}k\varepsilon\right]$$

also,

$k = \sqrt{\frac{\lambda-v+2C}{v}} = \left(\frac{\lambda+2C-v}{v}\right)^{\frac{1}{2}}.$

and

$\varepsilon = x - vt,\ \lambda = (p-q)$

therefore

$$U(x,t) = \frac{2}{\alpha}(v+q-p-C) + \frac{3}{\alpha}(\lambda+2C-v)sech^2\left[\frac{1}{2}\left(\frac{p-q+2C-v}{v}\right)^{\frac{1}{2}}(x-vt) + \vartheta\right]. \qquad (48)$$

In a special case where$C = v+q-p$,we obtain that

$$U(x,t) = \frac{3}{\alpha}(v+q-p)sech^2\left[\frac{1}{2}\left(\frac{v+q-p}{v}\right)^{\frac{1}{2}}(x-vt) + \vartheta\right].$$

The solution is therefore given as:

$$U(x,t) = 3\left\{\alpha(v+q-p)sech^2\left[\frac{1}{2}\left(\frac{v+q-p}{2v}\right)^{\frac{1}{2}}(x-vt) + \vartheta\right]\right\}^{-1}. \qquad (49)$$

But

$p = 2H(T-m)^{2H-1}$and $\ q = \frac{2r}{\sigma^2},$

Therefore equations (48) and (49) respectively become

$$U(x,t) = \frac{2}{\alpha}\left(v + \frac{2r}{\sigma^2} - 2H(T-m)^{2H-1} - C\right)$$

$$+\frac{3}{\alpha}(\lambda+2C-v)sech^2\left[\frac{1}{2}\left(\frac{\frac{2r}{\sigma^2}-2H(T-m)^{2H-1}+2C-v}{v}\right)^{\frac{1}{2}}(x-vt) + \vartheta\right], \quad (50)$$

and for the special case

$$U(x,t) = 3\left\{\alpha\left(v + \frac{2r}{\sigma^2} - 2H(T-m)^{2H-1}\right)sech^2\left[\frac{1}{2}\left(\frac{v+\frac{2r}{\sigma^2}-2H(T-m)^{2H-1}}{2v}\right)^{\frac{1}{2}}(x-vt) + \vartheta\right]\right\}^{-1}. \quad (51)$$

To consider the non-homogeneous case, we get from equation

$$\frac{\partial U}{\partial \tau}+\left[2H\left(T-\frac{2\tau}{\sigma^2}\right)^{2H-1}+\frac{2r}{\sigma^2}\right]\frac{\partial U}{\partial x}+\left[2H\left(T-\frac{2\tau}{\sigma^2}\right)^{2H-1}\right]\frac{\partial^2 v}{\partial x^2}=\frac{2rU}{\sigma^2}, \qquad (52)$$

which with

$$\psi(t)=\left[2H\left(T-\frac{2\tau}{\sigma^2}\right)^{2H-1}+\frac{2r}{\sigma^2}\right];\chi=\left[2H\left(T-\frac{2\tau}{\sigma^2}\right)^{2H-1}\right];\ g(t)=\frac{2rU}{\sigma^2},$$

reduces to

$$U_t+\chi U_{xx}+\psi(t)U_x=g(t), \qquad (53)$$

which we shall solve using algebraic method of the Modified Sine-Cosine.

The modified sine-cosine method is used to solve the non-homogeneous form of Benjamin-Bona-Mahony equation with time-dependent coefficients.

It of the form

$$U_t+\alpha U_{xxt}+[\beta(t)+\gamma(t)U]U_x=g(t)(54)$$

where α is a real constant and $\beta(t),\gamma(t)$ and $g(t)$ are functions depending on the variable t only. Numerous investigations have explored various forms of the BBM equation through diverse methodologies, with notable contributions from Benjamin, Bona, and Mahony (1972), Wazwaz (2005), Alquran (2012), Alquran and Al-Khaled (2011), Chen, Lai, and Qing (2007), and Abazari (2013). Applying the framework of equation (53), equation (54) undergoes modification as follows:

$$U_t+\frac{\alpha}{\chi}U_{xxt}+[\beta(t)+\gamma(t)U]U_x=g(t). \qquad (55)$$

where

$$\psi(t)=[\beta(t)+\gamma(t)U]$$

The Modified Sine-Cosine and the solution homogeneous equation

The enhanced sine-cosine method, an extension of the conventional sine-cosine method, incorporates advancements introduced by Tascan and Bekir (2009), Ali, Soliman, and Raslan (2007), as well as the contributions of Alquran and Qawasmeh (2013) and Alquran and Al-Khaled (2011). This method acknowledges and employs solutions in the form of:

$$u(x,t)=A(t)\cos^m(\mu\zeta),\zeta=x-c(t) \qquad (56)$$

and

$$u(x,t)=A(t)\sin^m(\mu\zeta),\zeta=x-c(t). \qquad (57)$$

For some parameters$A(t),\mu,m$ and $c(t)$ to be determined later whereμ is the wave number and $c(t)$ is the wave speed being a function of the time t. From (56), we have

$$u_t(x,t)=A'(t)\cos^m\left(\mu\left(x-c(t)\right)\right)+m\mu A(t)c'(t)\cos^{m-1}\left(\mu\left(x-c(t)\right)\right)\sin\left(\mu\left(x-c(t)\right)\right)$$

$$u_t(x,t)=-m\mu A(t)\cos^{m-1}\left(\mu\left(x-c(t)\right)\right)\sin\left(\mu\left(x-c(t)\right)\right),$$

$$u_{xxt}(x,t)=$$
$$m(m-1)\mu^2A'(t)\cos^{m-2}\left(\mu\left(x-c(t)\right)\right)-m^2\mu^2A'(t)\cos^m\left(\mu\left(x-c(t)\right)\right)-m^3\mu^3A(t)c'(t)\cos^{m-1}\left(\mu\left(x-c(t)\right)\right)\sin\left(\mu\left(x-c(t)\right)\right)+m(m-1)(m-2)\mu^3A(t)c'(t)\cos^{m-3}\left(\mu\left(x-c(t)\right)\right)\sin\left(\mu\left(x-c(t)\right)\right). \qquad (58)$$

Due to the duality relationship between sine and cosine functions, and without loss of generality, we forego the analysis argument concerning the solution presented in (57). Subsequently, by substituting (58) into the original partial differential equation (55), a trigonometric equation emerges, featuring terms of either $\cos^n(\mu\zeta)$or $\cos^n(\mu\zeta)\sin(\mu\zeta)$). The parameter n can be determined by comparing exponents. The necessity for the coefficient of $\cos^i(\mu\zeta)$or $\cos^i(\mu\zeta)\sin(\mu\zeta)$to vanish for all powers of i results in a system of algebraic equations involving the unknowns $A(t), \mu, m$and $c(t)$. The solution proposed in (56) promptly follows from this system.

Generalized Benjamin-Bona-Mahony

Consider the following non-homogeneous BBM

$$u_t + \alpha u_{xxt} + \beta(t)u_x + \gamma(t)uu_x = g(t) \quad (59)$$

First, we use the transformation

$$u(x,t) = w(x,t) + \mathrm{h}(t) \quad (60)$$

Substituting (60) in (59) yields

$$w_t + \mathrm{h}'(t) + \alpha w_{xxt} + \beta(t)w_x + \gamma(t)ww_x + \gamma(t)\mathrm{h}(t)w_x = g(t) \quad (61)$$

We require that

$\mathrm{h}'(t) = g(t)$ so that $\mathrm{h}(t) = \int g(t)dt$

Hence, the following homogeneous BBM equation is obtained

$$w_t + \alpha w_{xxt} + k(t)w_x + \gamma(t)ww_x = 0, \quad (62)$$

where

$$k(t) = \beta(t) + \gamma(t)\mathrm{h}(t). \quad (63)$$

Through the substitution of the cosine assumptions from (56) and (58) into (62), and subsequent comparison of exponents and collection of coefficients for $\cos^i(\mu\zeta)$or $\cos^i(\mu\zeta)\sin(\mu\zeta)$across all values of i, the resulting algebraic system is as follows:

$$0 = m + 2$$
$$0 = (m-1)(m-2)\mu^2\alpha(t)c'(t) - A(t)\gamma(t)$$
$$0 = A'(t)\left(1 - m^2\mu^2\alpha(t)\right)$$
$$0 = m(m-1)\mu^2\alpha(t)A'(t),$$
$$0 = c'(t) - k(t) - m^2\mu^2\alpha(t)c'(t) \quad (64)$$

Solving the above system yields

$$m = -2$$
$$A'(t) = 0$$
$$c'(t) = \frac{3k(t) + A(t)\gamma(t)}{3}$$
$$\mu = \pm\frac{\sqrt{A(t)}}{2\sqrt{\alpha}\sqrt{3\frac{k(t)}{\gamma(t)} + A(t)}}. \quad (65)$$

We obtain the following facts from (65)

1. $A(t)$ must be a constant i.e. $A(t) = A$
2. $c(t) = \int \frac{3k(t) + A\gamma(t)}{3} dt$
3. Since μ is constant from the cosine assumption, therefore $\frac{k(t)}{\gamma(t)}$ must be constant. Thus, $k(t)$ is a multiple of $\gamma(t)$.

Therefore from the foregoing we obtain the solution of the BBM (59) as

$$u(x,t) = A\sec^2\left[\frac{\sqrt{A(t)}[x - \int \frac{3k(t)+A\gamma(t)}{3}dt]}{2\sqrt{\alpha}\sqrt{3\frac{k(t)}{\gamma(t)} + A(t)}}\right] + \int g(t)dt\ . \quad (66)$$

In the context of various cases of the non-homogeneous BBM equation characterized by diverse time-dependent coefficients, the following discussion unfolds:

Case 1: In this case we consider

$$\alpha = 1, \beta(t) = 1, \gamma(t) = 1, g(t) = 0, A = 1$$

By the constraints on the coefficient functions we find that

$\mathrm{h}(t) = 0, k(t) - 1,\ \mu = \pm\frac{1}{4}.$ and $c(t) = \frac{4t}{3}$

and the solution of the BBM is

$u(x,t) = \sec^2\left(\frac{3x-4t}{12}\right)$(67)

Case 2: Considering

$$\alpha = 1, \beta(t) = 2t - t^2, \gamma(t) = t, g(t) = 1, A = 1$$

we find that

$$\mathrm{h}(t) = t, k(t) = 2t, \mu = \pm\frac{1}{2\sqrt{7}}, c(t) = \frac{7t^2}{6}$$

and obtaining the solution

$$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(x - \frac{7t^2}{6}\right)\right) + t\ . \quad (68)$$

Case 3: In the case where

$$\alpha = 1, \beta(t) = 2e^{-t} - 1, \gamma(t) = e^{-1}, g(t) = e^t, A = 1$$

we find that

$$h(t) = e^t, k(t) = 2e^{-t}, \mu = \pm\frac{1}{2\sqrt{7}}, c(t) = \frac{-7e^{-t}}{3}$$

and the solution is

$$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(x - \frac{7t^2}{3}\right)\right) + e^t. \quad (69)$$

Case 4: Considering

$$\alpha = 1, \beta(t) = 2\sin(t) + 1, \gamma(t) = \sin(t), g(t) = \csc(t)\cot(t), A = 1$$

we find that

$$h(t) = -\csc(t), k(t) = 2\sin(t),\ \mu = \pm\frac{1}{2\sqrt{7}}, c(t) = \frac{-7\cos(t)}{3}$$

giving the solution

$$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(x - \frac{7\cos(t)}{3}\right)\right) - \csc(t). \quad (70)$$

But$x = \ln(S/K)$

From the above we have the solutions to our problem in the respective circumstances are

Case 1

$$u(x,t) = \sec^2\left(\frac{3\ln(S/K) - 4t}{12}\right)$$

Case 2

$$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(\ln(S/K) - \frac{7t^2}{6}\right)\right) + t$$

Case 3

$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(\ln(S/K) - \frac{7t^2}{3}\right)\right) + e^t.$

Case 4

$$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(\ln(S/K) - \frac{7\cos(t)}{3}\right)\right) - \csc(t)$$

Case 5

However for the classical Black-Scholes;

$\frac{\partial V}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 V}{\partial S^2} + [r + (\mu - \alpha)]x\frac{\partial V}{\partial S} - rV = -\alpha x (71)$

$x > 0, t > 0, H \in (0,1), H = \frac{1}{2}$

Solving the nonhomogeneous part of (71);

Let $u_P = A + Bx$, $V_p' = BV_p'' = 0$, then

$[r + (\mu - \alpha)]Bx - r(A + Bx) = \alpha x,$

so that

$rBx + (\mu - \alpha)Bx - rA - rBx = \alpha x,$

but $A = 0$ and $B = \frac{\alpha}{\mu - \alpha}$ when the coefficients are compared

Then

$u_P = \frac{\alpha}{\mu - \alpha}x.$

$$u(x,t) = \sec^2\left(\frac{1}{2\sqrt{7}}\left(\ln(S/K) - \frac{7\cos(t)}{3}\right)\right) - \frac{\alpha}{\mu - \alpha}x$$

4. Conclusion

The exploration of pricing for addressing the challenges associated with risky assets and their derivatives is a fundamental aspect of the field of mathematical finance. Within this domain, the problem of option pricing holds particular significance.

This study delves into the modeling of the option pricing equation using fractional Brownian motion, leading to the derivation of a solution for the Regular Long Wave Equation in the context of the Fractional Black-Scholes Option Pricing Model.

Further the modified sine-cosine method, coupled with symbolic computation, which proves to be a robust approach for effectively addressing the proposed BBM equation featuring time-dependent variable coefficients which used to find Solitary wave solutions are successfully derived, subject to specific constraints on the coefficient functions. Additionally, the study provides geometric illustrations that elucidate the physical structure of the non-homogeneous BBM was used to find the solution to thenon-homogeneous case ofour problem

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