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Use of Quintic B-Spline in Approximating Solutions of a Class of Nonlinear Singular Boundary Value Problems that Arise in Various Models

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Abstract: This paper introduces the quintic B-spline collocation method to approximate the solution of a class of non-linear singular boundary value problems with Neumann and Robin boundary conditions that arise in various models. L'Hôpital's rule modifies the singularity of the differential equation, which is subsequently converted into a solved problem. The investigation of the quintic B-spline interpolation error yields a truncation error of order six and convergence of order four. The method is uniformly convergent across the entire domain for the exact solution. Four broadly applicable problems have been solved to illustrate the accuracy of the proposed approach and numerical results have been compared to the exact solution at the same number of mesh points. The accuracy of the proposed approach is evaluated by determining the maximum absolute error and absolute residual error. Together with the exact solution and previous findings in the literature, the proposed approach yields better numerical results and it significantly reduces the computational cost of solving nonlinear singular boundary value problems.In addition to handling singularities, the proposed method manages larger domains and offers solutions for problems with strong non-linearity. Solving and analyzing nonlinear singular boundary value problems can provide insights into complex physical phenomena and aid in understanding the behavior of the model exhibiting nonlinear and singular characteristics.

Key words: Emden–Fowler equation; Quintic B-spline;Robin boundary conditions; singular value problems; uniformly convergent

1. Introduction

Differential equations are the outcome of the relationship that can be established between the laws governing the quantities involved in a physical phenomenon and their derivatives. In science and engineering, the majority of differential equations with boundary conditions cannot be solved analytically; as a result, numerical approximation techniques are needed. Solving singular differential equations makes it even more difficult. The solution's behavior close to the singular points can be extremely complex and challenging to understand. It is also essential to comprehend how the solution behaves close to the singular points in order to derive accurate solutions and capture significant aspects of the equations. Nonlinear interactions between singular differential equations often exhibit complex behavior and can be challenging to understand and study (Larry 2007; Boor 1978; Burden and Faires 2003). In recent years, the study of nonlinear singular boundary value problems has attracted a lot of attention and developed in science and engineering research. Solving and analyzing nonlinear singular boundary value problems can provide insights into complex physical phenomena and aid in understanding the behavior of the model exhibiting nonlinear and singular characteristics.

In this work, the class of nonlinear singular boundary value problems of the form that are considered is as follows.

$$(t^{\alpha}u'(t))' = \mathrm{t}^{\alpha} f(\mathrm{t},\mathrm{u(t)}), \qquad 0 < t \le 1\,,\ \ \alpha \ge 1\,,\ \ \mathrm{u} \in (0,\infty) \tag{1}$$

subject to the Neumann and Robin boundary conditions

$$u'(0) = 0, \tag{2}$$

and

$$a_1 u(1) + a_2 u'(1) = \mathrm{b}, \tag{3}$$

where $a_1 > 0,\ a_2 \ge 0,\ b$ are any finite real constants and assumed that the nonlinear function $f(\mathrm{t},\mathrm{u(t)})$ and $\frac{\partial f}{\partial u}$ are continuous for every (t,u) and the condition $\frac{\partial f}{\partial u} \ge 0$ be satisfied. This class of problems has numerous applications by varying the function $f(\mathrm{t},\mathrm{u(t)})$ and the value α a_1, a_2 and b such as distribution of heat sources in the human head, oxygen diffusion in a spherical cell, isothermal gas sphere, and radial stress on a rotationally symmetric shallow membrane cap, electro-hydrodynamics, bridge construction, astronomy, fluid flow, and other complex problems (Lin 2021; Lin 1976; Parand and Delkhosh 2017; Irena et al. 2007; Bobisud 1990). Flesch (1975) used a second-order singular differential equation to predict the link between the temperature distribution in the human head and the surrounding environment. It is difficult to find exact solutions of equation (1) because of the singular behavior at the origin and non-linearity. Researchers have applied several methods to solve such problems (Hikmot, Nazan, and Mehmet 2009; Kadalbajoo and Kumar 2007; Umesh 2021; Niu et al. 2018; Pandey and Tomar 2021; Pandey and Verma 2008; Rashidinia, Mahmoodi, and Ghasemi 2007; Roul and Goura 2020; Roul,Kumari, and Goura 2021; Roul, Prasad, and Agarwal 2022). Several semi-analytical approaches, including the variational iteration method, the Adomian decomposition method, and the optimal homotopy analysis method, have produced closed-form solutions of these problems with specific boundary conditions (Singh and Kumar 2014a; Singh 2018; Wazwaz 2011).

To solve singular boundary value problems, Kanth and Reddy (2003) re-approximated the central difference approximation and introduced a fourth order finite difference technique. Cohen and Jones (1974) used the finite difference deferred correction technique to handle second order singular boundary value problems in the interval [0,1], they disregarded the singularity's impact on the solution in the vicinity of the singular point. Pandey and Singh (2004) presented a finite difference approach based on a uniform mesh and it was demonstrated that the method is of second-order accuracy under very general conditions. A numerical method for determining pointwise bounds for the solution of a class of nonlinear boundary-value problems that arise in physiology was introduced by Asaithambi and Garner (1989). Singh and Kumar (2014b) presented a novel approach to nonlinear singular boundary value problems that is based on Green's function and the Adomian decomposition method. Singh, Kumar, and Nelakanti (2013) used Green's function and improved decomposition method for solving singular boundary value problems. Khuri and Sayfy(2010) introduced a novel numerical approach that connected the modified Adomian decomposition method with the B-spline collocation scheme. Because of their rich geometrical features, accuracy, and smoothness, B-splines have been widely employed for solving boundary value problems throughout the past three decades. A numerical technique for calculating approximations to the solution of nonlinear singular boundary value problems related to physiological research was developed by Hikmot, Nazan, and Mehmet (2009) using B-spline functions and the B-spline approximation was used to treat the boundary value problem after the original differential equation was altered at the unique point. Maria and Dambaru(2015) investigated the approximate solutions to second order linear boundary value problems using cubic B-splines. Kanth and Bhattacharya (2006) used B-spline functions to solve two-point boundary value problems with a singularity at $x = 0$ after employing quasi-linearization techniques to reduce the non-linear problem into a series of linear problems and then altering the resulting sets of differential equations around the singular point. Kadalbajoo and Aggarwal (2003) employed Chebyshev economization in the vicinity of the unique point before deriving a boundary condition at a point in the vicinity of the singularity in order to eliminate the singularity for homogeneous and linear singular boundary value problems. Goh, Majid, and Ismail (2011) used extended uniform B-spline functions to investigate the approximate solution of linear singular boundary value problems. Recently, Alam and Khan (2024); Isadi and Saini (2015) employed novel techniques to tackle singular boundary value problems.

Despite the many advantages of these numerical methods, a thorough analysis of existing approaches in the literature review revealed that they have a significant drawback, including the need for a significant amount of computational work since the domain must be discretized and the considered problem must be linearized. It may also be challenging to handle singularities using techniques like the cubic spline method and the finite difference approach, fails to provide solutions for problems with strong nonlinearity, such as variational iteration method, fails for larger domains, and requires additional effort to obtain unknown parameters and take additional terms from the approximate solution, such as Adomian decomposition. This motivation leads to an accurate numerical technique to overcome these limitations and offer more accurate answers to this class of problems so that a higher order B-spline approach is suitable for solving these kinds of problems. Accordingly, quintic B-spline collocation method is proposed to tackle the problems. The advantage of quintic B-splines is that the polynomials are always of degree five and with lower computational cost, while in the case of other polynomials; the degree is quite high which depends on the number of subintervals. Moreover, it can also be found the approximate values from first derivative to fourth derivative at the knots.

In this paper, a quintic B-spline solution method is used to solve a class of nonlinear singular boundary value problems of equations (1), (2) and (3). L'Hôpital's rule modifies the singularity of the differential equation, which is subsequently converted into a solved problem. Examples have been solved to illustrate the accuracy of the proposed approach and numerical results have been compared to the exact solution if available.

2. The quintic B-spline collocation method

A knot sequence t is a non-decreasing sequence of real numbers, $t := \{t_i\}_{i=1}^{n} = \{t_1 \le t_2 \le \ldots \le t_n\}$, $n \in \mathrm{N}$. B-spline of degree p over the knot sequence t can be defined if $n \ge p+2$.

Definition. Suppose for the non-negative integer p and some integer i that $t_i \le t_{i+1} \le \ldots \le t_{i+p+1}$ are $p+2$ real numbers taken from a knot sequence t. The i-th B-spline $B_i^p : \mathrm{R} \to \mathrm{R}$ of degree p is defined as

$$B_i^p(\mathrm{x}) = \begin{cases} 0, & \text{if } t_{i+p+1} = t_i \\ \omega_i^p(\mathrm{x})\,\mathrm{B}_i^{p-1}(\mathrm{x}) + (1-\omega_{i+1}^p(\mathrm{x}))\,\mathrm{B}_{i+1}^{p-1}(\mathrm{x}), & \text{if } t_{i+p+1} \ne t_i \end{cases} \tag{4}$$

starting with $B_i^0(\mathrm{x}) = \begin{cases} 1, & \text{if } x \in [t_i, t_{i+1}), \\ 0, & \text{otherwise} \end{cases}$ where $\omega_i^p(\mathrm{x}) = \begin{cases} \dfrac{x - t_i}{t_{i+p} - t_i}, & \text{if } t_{i+p+1} \ne t_i, \\ 0, & \text{if } t_{i+p+1} = t_i \end{cases}$

$i = 0, \pm 1, \pm 2, \ldots$, and x is a parameter in [0, 1].

$0 \le i \le n-p-1,\ 1 \le p \le n-1$

Convention: (i) $\dfrac{m}{0} = 0$ where $m \in R$

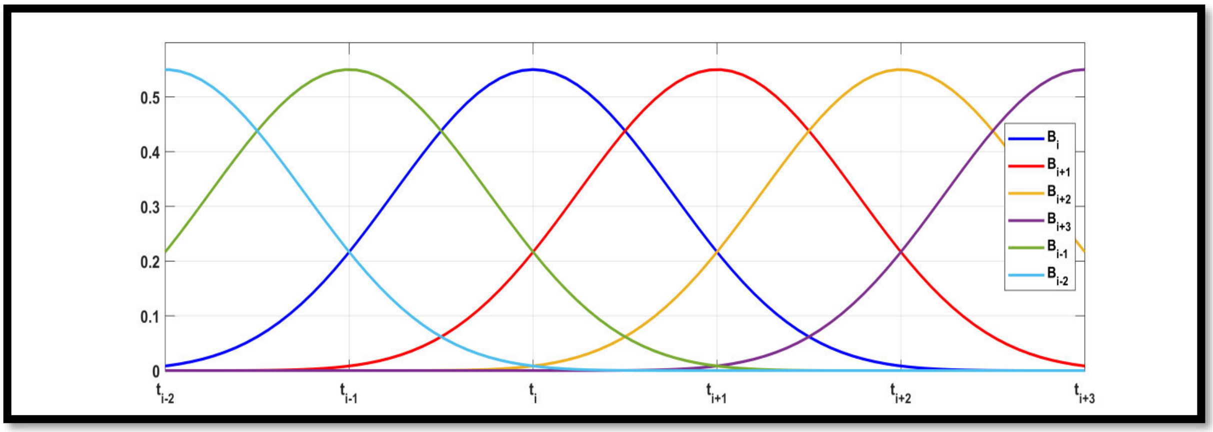
(ii) Some of the subscripts will be dropped and the B-spline written as either B_i^p or B_i where there is no possibility of ambiguity.

Let $\Delta_n : 0 = t_0 < t_1 < t_2 < \ldots < t_n = 1$ be a partition of [0,1] with the mesh size $h = \dfrac{1}{n}$ by the knots $t_i = ih,\ i = 0,1,2,\ldots,n$. After including four additional knots outside the domain of the partition and then the partition becomes $\Delta_n : t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < t_{n+2}$

We apply the recursion formula (4) to get the quintic B-spline B_i, $i = -2, -1, 0, 1, \ldots, n, n+1, n+2$ at the knots t_i defines over [0,1]as follows:

$$B_i(x) = \frac{1}{120h^5} \begin{cases} (x - t_{i-3})^5, & t_{i-3} \le x \le t_{i-2}, \\ (x - t_{i-3})^5 - 6(x - t_{i-2})^5, & t_{i-2} \le x \le t_{i-1}, \\ (x - t_{i-3})^5 - 6(x - t_{i-2})^5 + 15(x - t_{i-1})^5, & t_{i-1} \le x \le t_i, \\ (t_{i+3} - x)^5 - 6(t_{i+2} - x)^5 + 15(t_{i+1} - x)^5, & t_i \le x \le t_{i+1}, \\ (t_{i+3} - x)^5 - 6(t_{i+2} - x)^5, & t_{i+1} \le x \le t_{i+2}, \\ (t_{i+3} - x)^5, & t_{i+2} \le x \le t_{i+3}, \\ 0, & \text{elsewhere.} \end{cases} \tag{5}$$

Figure 1: B_i's in the interval [0,1] and h=1/6



2.2 Properties of quintic B-spline

A number of significant characteristics may be inferred from Figure 1 and the recurrence defining B_i:

(a) Nonnegativity:

$$B_i(x) \geq 0, \quad x \in R, \quad \text{and} \quad B_i(x) > 0, \quad x \in (t_{i-3}, t_{i+3})$$

(b) Local support:

$$B_i(x) = 0, \quad x \notin [t_{i-3}, t_{i+3})$$

(c) Normalization:

$$\sum_{i=-2}^{n+2} B_i(x) = 1, \quad x \in [0,1]$$

(d) Translation Invariance:

$$B_i(ax+b) = B_i(x), \; a, b \in R, \; a \neq 0, \text{ where}$$

$$at + b = (at_{i-3} + b, \ldots, at_{i+3} + b).$$

(e) Differentiation Formula: The m-th derivative of a B-spline:

$$D^{\mathrm{m}}\left(\sum_i d_i\, B_i^5\right) = \sum_i d_i^{m+1} B_i^{5-m}, \text{where}$$

$$d_r^{m+1} = \begin{cases} d_r, & m = 0 \\ \dfrac{d_r^m - d_{r-1}^m}{(t_{r+5-m} - t_r)/(5-\mathrm{m})}, & m > 0 \end{cases}$$ and fractions with zero denominator have value zero.

3. The numerical approach for singular BVPS

For solving Eq. (1)-Eq. (3), we first modify Eq. (1) at the singular point $x=0$ using L'Hopital's rule and then, transform the given problem Eq. (1)-Eq. (3) into the following form:

$$u''(\mathrm{t}) + \mathrm{h}(\mathrm{t})\,\mathrm{u}'(\mathrm{t}) = \mathrm{g}(\mathrm{t}, \mathrm{u}(\mathrm{t}))$$

$$u'(0) = 0, \qquad (6)$$

$$a_1 u(1) + a_2 u'(1) = \mathrm{b},$$

where $h(\mathrm{t}) = \begin{cases} 0, & \mathrm{t} = 0 \\ \dfrac{\alpha}{t}, & t \neq 0 \end{cases}$ and $\mathrm{g}(\mathrm{t}, \mathrm{u}(\mathrm{t})) = \begin{cases} \dfrac{f(\mathrm{t}, \mathrm{u}(\mathrm{t}))}{\alpha + 1}, & t = 0, \\ f(\mathrm{t}, \mathrm{u}(\mathrm{t})), & \mathrm{t} \neq 0 \end{cases}$

The collocation method with quintic B-spline as basis functions is used to solve the boundary value problems Eq. (6). Let

$$Q(t) = \sum_{i=-2}^{n+2} d_i B_i(t) \tag{7}$$

be an approximate solution to the exact solution u(x) to Eq. (6) where d_i are the unknown constants to be determined from the boundary conditions and collocation points from the differential equation and B_i are quintic B-spline functions. The internal mesh points $t_1, t_2, \dots, t_{n-1}$ are chosen as the collocation points in the proposed method. Eq. (7) satisfies and can be simplified to

$$u(t_i) \cong Q(t_i) = d_{i-2} B_{i-2}(t_i) + d_{i-1} B_{i-1}(t_i) + d_i B_i(t_i) + d_{i+1} B_{i+1}(t_i) + d_{i+2} B_{i+2}(t_i) \tag{8}$$

$$u'(t_i) \cong Q'(t_i) = d_{i-2} B'_{i-2}(t_i) + d_{i-1} B'_{i-1}(t_i) + d_i B'_i(t_i) + d_{i+1} B'_{i+1}(t_i) + d_{i+2} B'_{i+2}(t_i) \tag{9}$$

$$u''(t_i) \cong Q''(t_i) = d_{i-2} B''_{i-2}(t_i) + d_{i-1} B''_{i-1}(t_i) + d_i B''_i(t_i) + d_{i+1} B''_{i+1}(t_i) + d_{i+2} B''_{i+2}(t_i) \tag{10}$$

$$u'''(t_i) \cong Q'''(t_i) = d_{i-2} B'''_{i-2}(t_i) + d_{i-1} B'''_{i-1}(t_i) + d_i B'''_i(t_i) + d_{i+1} B'''_{i+1}(t_i) + d_{i+2} B'''_{i+2}(t_i) \tag{11}$$

$$i = 0, 1, 2, \dots, n$$

Substituting Eq. (8)-Eq. (10) into Eq. (6), we get

$$d_{i-2} B''_{i-2}(t_i) + d_{i-1} B''_{i-1}(t_i) + d_i B''_i(t_i) + d_{i+1} B''_{i+1}(t_i) + d_{i+2} B''_{i+2}(t_i) +$$
$$h(t_i)[d_{i-2} B'_{i-2}(t_i) + d_{i-1} B'_{i-1}(t_i) + d_i B'_i(t_i) + d_{i+1} B'_{i+1}(t_i) + d_{i+2} B'_{i+2}(t_i)] = g(t_i, u(t_i))$$

This implies

$$\begin{aligned} &d_{i-2}[B''_{i-2}(t_i) + h(t_i) B'_{i-2}(t_i)] + d_{i-1}[B''_{i-1}(t_i) + h(t_i) B'_{i-1}(t_i)] + d_i[B''_i(t_i) + h(t_i) B'_i(t_i)] \\ &+ d_{i+1}[B''_{i+1}(t_i) + h(t_i) B'_{i+1}(t_i)] + d_{i+2}[B''_{i+2}(t_i) + h(t_i) B'_{i+2}(t_i)] = g(t_i, u(t_i)) \end{aligned} \tag{12}$$

Using the properties of quintic B-spline functions, we get

$$\begin{array}{lllll} B_{i-2}(t_i) = \frac{1}{120} & B'_{i-2}(t_i) = \frac{1}{24h} & B''_{i-2}(t_i) = \frac{1}{6h^2} & B'''_{i-2}(t_i) = \frac{1}{2h^3} & B^{(4)}_{i-2}(t_i) = \frac{2}{2h^4} \\ B_{i-1}(t_i) = \frac{26}{120} & B'_{i-1}(t_i) = \frac{10}{24h} & B''_{i-1}(t_i) = \frac{2}{6h^2} & B'''_{i-1}(t_i) = -\frac{2}{2h^3} & B^{(4)}_{i-1}(t_i) = -\frac{4}{h^4} \\ B_i(t_i) = \frac{66}{120} & B'_i(t_i) = 0 & B''_i(t_i) = -\frac{6}{6h^2} & B'''_i(t_i) = 0 & B^{(4)}_i(t_i) = \frac{6}{h^4} \\ B_{i+1}(t_i) = \frac{26}{120} & B'_{i+1}(t_i) = -\frac{10}{24h} & B''_{i+1}(t_i) = \frac{2}{6h^2} & B'''_{i+1}(t_i) = \frac{2}{2h^3} & B^{(4)}_{i+1}(t_i) = \frac{-4}{h^3} \\ B_{i+2}(t_i) = \frac{1}{120} & B'_{i+2}(t_i) = -\frac{1}{24h} & B''_{i+2}(t_i) = \frac{1}{6h^2} & B'''_{i+2}(t_i) = -\frac{1}{2h^3} & B^{(4)}_{i+2}(t_i) = \frac{1}{h^4} \end{array} \tag{13}$$

Substituting Eq. (13) in Eq. (12), we obtain

$$\begin{aligned} &d_{i-2}[4 + h(t_i)h] + d_{i-1}[8 + 10h(t_i)h] + d_i[-24] + d_{i+1}[8 - 10h(t_i)h] + \\ &\qquad d_{i+2}[4 - h(t_i)h] = 24h^2 g(t_i, u(t_i)),\ i = 0, 1, 2, \dots, n \end{aligned} \tag{14}$$

For $i = 0$, we have $h(t_i) = 0$. Hence,

$$d_{-2} + 2d_{-1} - 6d_0 + 2d_1 + d_2 = 6h^2 f_0, \text{ where } f_0 = \frac{f(t_0, u(t_0))}{\alpha + 1} \tag{14a}$$

For $i = 1, 2, \dots, n$, we have $h(t_i) = \frac{\alpha}{t_i}$. Hence,

$$(4t_i+\alpha h)d_{i-2}+(8t_i+10\alpha h)d_{i-1}-24d_i+(8t_i-10\alpha h)d_{i+1}+$$
$$(4t_i-\alpha h)d_{i+2}=24t_i h^2 f_i, \text{ where } f_i=f(t_i,u(t_i)) \tag{14b}$$

Using the first boundary condition, Eq. (9) and Eq. (13), we get

$$Q'(t_0)=d_{i-2}B'_{i-2}(t_0)+d_{i-1}B'_{i-1}(t_0)+d_i B'_i(t_0)+d_{i+1}B'_{i+1}(t_0)+d_{i+2}B'_{i+2}(t_0)=0$$

This implies

$$d_{-2}+10d_{-1}-10d_1-d_2=0 \tag{15}$$

Using the second boundary condition, Eq. (8), Eq. (9) and Eq. (13), we get

$$a_1[d_{n-2}B_{n-2}(t_n)+d_{n-1}B_{n-1}(t_n)+d_n B_n(t_n)+d_{n+1}B_{n+1}(t_n)+d_{n+2}B_{n+2}(t_n)]$$
$$+a_2[d_{n-2}B'_{n-2}(t_n)+d_{n-1}B'_{n-1}(t_n)+d_n B'_n(t_n)+d_{n+1}B'_{n+1}(t_n)+d_{n+2}B'_{n+2}(t_n)]=b$$

This implies

$$(a_1h+5a_2)d_{n-2}+(26a_1h+50a_2)d_{n-1}+(66a_1h)d_n+(26a_1h-50a_2)d_{n+1}$$
$$+(a_1h-5a_2)d_{n+2}=120bh \tag{16}$$

Now it has been found $n+3$ equations, but $n+5$ unknowns. So, two equations are needed to determine a unique solution of the system Eq. (14), Eq. (15) and Eq. (16). When we differentiate Eq. (6) with respect to t three times, we obtain

$$u'''(t)+h(t)u''(t)+h'(t)u'(t)=u'(t)\frac{dg(t,u(t))}{dt} \tag{17}$$

where $h(t)=\begin{cases}0, & t=0\\ \dfrac{\alpha}{t}, & t\neq 0\end{cases}$ $\qquad h'(t)=\begin{cases}0, & t=0\\ -\dfrac{\alpha}{t^2}, & t\neq 0\end{cases}$

At $t=0$, $h(t)u''(t)+h'(t)u'(t)$ is indeterminate form. Hence, by L'Hopital's rule, its limit as $x\to 0$ is $\frac{\alpha}{2}u'''$. Therefore, Eq. (17) becomes

$$\left(1+\frac{\alpha}{2}\right)u'''(t_0)=0 \tag{18}$$

Using Eq. (11), Eq. (12), and Eq. (18)

$$\left(1+\frac{\alpha}{2}\right)[d_{-2}B'''_{-2}(t_0)+d_{-1}B'''_{-1}(t_0)+d_0 B'''_0(t_0)+d_1 B'''_1(t_0)+d_2 B'''_2(t_0)]=0$$

This implies

$$d_{-2}-2d_{-1}+2d_1-d_2=0 \tag{19}$$

At $t=t_n$, Eq. (17) becomes

$$u'''(t_n)+h(t_n)u''(t_n)+h'(t_n)u'(t_n)=u'(t)\left.\frac{dg(t,u(t))}{du}\right|_{t=t_n}$$

$$Q'''(t_n)+h(t_n)Q''(t_n)+h'(t_n)Q'(t_n)=u'(t)\left.\frac{dg(t,u(t))}{du}\right|_{t=t_n}$$

$$
\mathrm{d}_{n-2}\,B'''_{n-2}(\mathrm{t}_n)+\mathrm{d}_{n-1}\,B'''_{n-1}(\mathrm{t}_n)+\mathrm{d}_n\,B'''_{n}(\mathrm{t}_n)+\mathrm{d}_{n+1}\,B'''_{n+1}(\mathrm{t}_n)+\mathrm{d}_{n+2}\,B'''_{n+2}(\mathrm{t}_n)+
$$

$$
\mathrm{h}(\mathrm{t}_n)[\mathrm{d}_{n-2}\,B''_{n-2}(\mathrm{t}_n)+\mathrm{d}_{n-1}\,B''_{n-1}(\mathrm{t}_n)+\mathrm{d}_n\,B''_{n}(\mathrm{t}_n)+\mathrm{d}_{n+1}\,B''_{n+1}(\mathrm{t}_n)+\mathrm{d}_{n+2}\,B''_{n+2}(\mathrm{t}_n)]+
$$

$$
\mathrm{h}'(\mathrm{t}_n)[\mathrm{d}_{n-2}\,B'_{n-2}(\mathrm{t}_n)+\mathrm{d}_{n-1}\,B'_{n-1}(\mathrm{t}_n)+\mathrm{d}_n\,B'_{n}(\mathrm{t}_n)+\mathrm{d}_{n+1}\,B'_{n+1}(\mathrm{t}_n)+\mathrm{d}_{n+2}\,B'_{n+2}(\mathrm{t}_n)]=
$$

$$
\mathrm{u}'(\mathrm{t})\left.\frac{dg(\mathrm{t},\mathrm{u}(\mathrm{t}))}{du}\right|_{t=t_n}
$$

This implies

$$
(12\,\mathrm{t}_n^2+4\alpha\,\mathrm{h}t_n-\alpha\,\mathrm{h}^2)\,\mathrm{d}_{n-2}+(-24\,\mathrm{t}_n^2+8\alpha\,\mathrm{h}t_n-10\alpha\,\mathrm{h}^2)\,\mathrm{d}_{n-1}+
$$

$$
(-24\alpha t_n h)\,\mathrm{d}_n+(24\,\mathrm{t}_n^2+8\alpha\,\mathrm{h}t_n+10\alpha\,\mathrm{h}^2)\,\mathrm{d}_{n+1}+(-12\,\mathrm{t}_n^2+4\alpha\,\mathrm{h}t_n+\alpha\,\mathrm{h}^2)\,\mathrm{d}_{n+2} \quad (20)
$$

$$
=24\,\mathrm{h}^3\mathrm{t}_n^2\,\xi_n, \qquad \text{where } \xi_n=\mathrm{u}'(\mathrm{t}_n)\frac{df(\mathrm{t}_n,\mathrm{u}(\mathrm{t}_n))}{du}
$$

It is found a system of n+5 equations in n+5 unknowns since we have determined all of the constant coefficients in equations (14), (15), (16), (19), and (20). This system is represented in matrix form:

$$
AX=B \quad (21)
$$

where $X=[\mathrm{d}_{-2},\ \mathrm{d}_{-1},\ \mathrm{d}_0,\ \mathrm{d}_1,\ldots,\ \mathrm{d}_n,\ \mathrm{d}_{n+1},\ \mathrm{d}_{n+2}]^T$

$$
B=[0,\ 0,\ 6h^2 f_0,\ \ 24t_1h^2 f_1,\ \ldots,\ \ 24t_nh^2 f_n,\ \ 24t_n^2h^3\xi_n,\ \ 120bh]^T
$$

$$
A=\begin{bmatrix}
1 & 10 & 0 & -10 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
1 & -2 & 0 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2 & -6 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & p_1 & q_1 & -24 & r_1 & s_1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & p_2 & q_2 & -24 & r_2 & s_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & p_{n-1} & q_{n-1} & -24 & r_{n-1} & s_{n-1} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & p_n & q_n & -24 & r_n & s_n\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \beta_1 & \beta_2 & \beta_3 & \beta_3 & \beta_4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5
\end{bmatrix}
$$

where

$$
p_i=4t_i+\alpha h,\ q_i=8t_i+10\alpha h,\ r_i=8t_i-10\alpha h,\ s_i=4t_i-\alpha h,\ \beta_1=12t_n^2+4\alpha h t_n-\alpha h^2
$$

$$
\beta_2=-24t_n^2+8\alpha h t_n-10\alpha h^2,\ \ \beta_3=-24\alpha t_n h,\ \ \beta_4=24t_n^2+8\alpha h t_n+10\alpha h^2
$$

$$
\beta_5=-12t_n^2+4\alpha h t_n+\alpha h^2,\ \ \alpha_1=a_1h+5a_2,\ \ \alpha_2=26a_1h+50a_2
$$

$$
\alpha_3=66a_1h,\ \ \alpha_4=26a_1h-50a_2,\ \ \alpha_5=a_1h-5a_2
$$

Once the values of d_{-2}, d_{-1}, d_0, $\mathrm{d}_1,\ldots,$ d_n, d_{n+1}, and d_{n+2} have been determined from the system of nonlinear equation (21) and then substitute these values into equation (7). Equation (7) can give an accurate approximation that is guaranteed to match it, if an exact solution is available.

4. Computation of error and order of convergence

This section outlines a procedure for calculating the quintic B-spline method's truncation error over $0 \le t \le 1$. The function u(t) with continuous derivatives throughout the entire range is taken into consideration. The following relations are obtained usingthe quintic B-spline approximations and equations (8)-(11) and (13) (Mishra and Saini 2015; Xu and Lang 2014):

$$Q(t_i)=\frac{1}{120}d_{i-2}+\frac{26}{120}d_{i-1}+\frac{66}{120}d_i+\frac{26}{120}d_{i+1}+\frac{1}{120}d_{i+2}$$
$$Q'(t_i)=\frac{1}{h}\left[\frac{1}{24}d_{i-2}+\frac{10}{24}d_{i-1}-\frac{10}{24}d_{i+1}-\frac{1}{24}d_{i+2}\right]$$
$$Q''(t_i)=\frac{1}{h^2}\left[\frac{1}{6}d_{i-2}+\frac{2}{6}d_{i-1}-d_i+\frac{2}{6}d_{i+1}+\frac{1}{6}d_{i+2}\right]$$
$$Q'''(t_i)=\frac{1}{h^3}\left[\frac{1}{2}d_{i-2}-d_{i-1}+d_{i+1}-\frac{1}{2}d_{i+2}\right]$$
$$Q^{(4)}(t_i)=\frac{1}{h^4}\left[d_{i-2}-4d_{i-1}+6d_i-4d_{i+1}+d_{i+2}\right]$$
$$i=0,1,2,\ldots,n \qquad (22)$$

Let $u(t)$ have continuous derivatives for all $t\in[0,1]$. Using Equation (22) the following relationship can be obtained:

$$Q'(t_{i-2})+26Q'(t_{i-1})+66Q'(t_i)+26Q'(t_{i+1})+Q'(t_{i+2})=\frac{1}{h}\left[5u(t_{i-2})+50u(t_{i-1})-50u(t_{i+1})-5u(t_{i+2})\right]$$
$$Q''(t_{i-2})+26Q''(t_{i-1})+66Q''(t_i)+26Q''(t_{i+1})+Q''(t_{i+2})=$$
$$\frac{1}{h^2}\left[20u(t_{i-2})+40u(t_{i-1})-120u(t_i)+40u(t_{i+1})+20u(t_{i+2})\right]$$
$$Q'''(t_{i-2})+26Q'''(t_{i-1})+66Q'''(t_i)+26Q'''(t_{i+1})+Q'''(t_{i+2})=$$
$$\frac{1}{h^3}\left[60u(t_{i-2})-120u(t_{i-1})+120u(t_{i+1})-60u(t_{i+2})\right] \qquad (23)$$
$$Q^{(4)}(t_{i-2})+26Q^{(4)}(t_{i-1})+66Q^{(4)}(t_i)+26Q^{(4)}(t_{i+1})+Q^{(4)}(t_{i+2})=$$
$$\frac{1}{h^4}\left[120u(t_{i-2})-480u(t_{i-1})+720u(t_{i-1})-480u(t_{i+1})+120(t_{i+2})\right]$$

Using shift operator $E(u(t_i))=u(t_{i+1})$, equation (23) can be expressed as

$$\left(E^{-2}+26E^{-1}+66+26E+E^2\right)Q'(t_i)=\frac{1}{h}\left[5E^{-2}+50E^{-1}-50E-5E^2\right]u(t_i)$$
$$\left(E^{-2}+26E^{-1}+66+26E+E^2\right)Q''(t_i)=\frac{1}{h^2}\left[20E^{-2}+40E^{-1}-120+40E+20E^2\right]u(t_i)$$
$$\left(E^{-2}+26E^{-1}+66+26E+E^2\right)Q'''(t_i)=\frac{1}{h^3}\left[60E^{-2}-120E^{-1}+120E-60E^2\right]u(t_i) \qquad (24)$$
$$\left(E^{-2}+26E^{-1}+66+26E+E^2\right)Q^{(4)}(t_i)=\frac{1}{h^4}\left[120E^{-2}-480E^{-1}+720-480E+120E^2\right]u(t_i)$$

Using the Operator $E=e^{hD}$, where $D=\frac{d}{dx}$ in an expansion form to the power of hD, we obtain

$$Q'(t_i)=u'(t_i)+o(h^6)$$
$$Q''(t_i)=u''(t_i)+\frac{1}{720}h^4u^{(6)}(t_i)+o(h^6)$$
$$Q'''(t_i)=u'''(t_i)-\frac{1}{240}h^4u^{(7)}(t_i)+o(h^6)$$
$$Q^{(4)}(t_i)=u^{(4)}(t_i)-\frac{1}{12}h^2u^{(6)}(t_i)+\frac{1}{240}h^4u^{(8)}(t_i)+o(h^6)$$

(25)

Theorem. Let $u(t) \in C^{\infty}[0,1]$ and $Q(t)$ be the approximate solution of $u(t)$. Then the global error bound is $\left\| u^{(r)}(t) - Q^{(r)}(t) \right\|_{\infty} = o\left(h^{6-r}\right)$, for r=0,1,2,3,4.

At the i^{th} knot, the error term is defined as $e^{(n)}(t_i) = Q^{(n)}(t_i) - u^{(n)}(t_i)$. The Taylor series expansion of the error term $e(t_i + \phi h)$ yields the following results when we use equation (25):

$e(t_i + \phi h) = e(t_i) + O(h^6), \qquad 0 \le \phi \le 1.$

5. Numerical Examples and Discussion

To illustrate the accuracy of the proposed method, four examples are taken into consideration, with the first two having exact solutions and the final two lacking any. The numerical results obtained are compared with the results obtained using another method. To assess the accuracy of the proposed method, the accuracy of the proposed scheme shown is confirmed using the maximum absolute error $L_{\infty} = \left\| Q_i - u_i \right\|_{\infty} = \max_i \left| Q_i - u_i \right|$ where Q_i and u_i are approximate and exact solutions at each knots and if the exact solution is not known, we compute the absolute residual error $R = \left| (t^{\alpha} Q'(t))' - t^{\alpha} f(t, Q(t)) \right|$, $0 < t \le 1$. MATLAB is used to generate the results.

Example 1. Consider the Emden–Fowler equation of the first kind (Lin 1976; Singh 2018)

$$(t^2 u'(t))' = t^2 u^5(t), \quad 0 < t < 1$$
$$u'(0) = 0, \quad u(1) = 0.8660254. \tag{26}$$

This problem is a particular of Eq. (1) - Eq. (3). That means $\alpha=2$, $a_1=1$, $a_2=0$, $b = 0.8660254$ and $f(t, u(t)) = u^5(t)$.

The exact solution of Eq. (26) is

$$u(t) = \left(\frac{3}{3+t^2} \right)^{1/2}.$$

The approximate solution using the proposed method and the absolute errors from the exact solution at each knots for this problem are tabulated in Table 1. The maximum absolute error using n=10 in [27] is 4.8643×10^{-5} but in the proposed method the maximum absolute error is $3.9153665262\times10^{-5}$ and Figure 2 shows the graphs of exact and approximate solutions.

Table 1: Quintic B-spline results and absolute errors for Example 1

t	Proposed Method	Exact	Absolute Error
0.0	0.9999982200000000	1.0000000000000000	0.0000017800000000
0.1	0.9983339588745640	0.9983374884595827	0.0000035295850187
0.2	0.9933959350592531	0.9933992677987828	0.0000033327395298
0.3	0.9853253627977669	0.9853292781642932	0.0000039153665262
0.4	0.9743525690790820	0.9743547036924464	0.0000021346133644
0.5	0.9607676052402520	0.9607689228305228	0.0000013175902709
0.6	0.9449104580998380	0.9449111825230680	0.0000007244232301
0.7	0.9271433547564660	0.9271455408231196	0.0000021860666536
0.8	0.9078387358752100	0.9078412990032037	0.0000025631279936

0.9	0.8873536169598070	0.8873565094161138	0.0000028924563068
1.0	0.8660222336193431	0.8660254037844386	0.0000031701650955

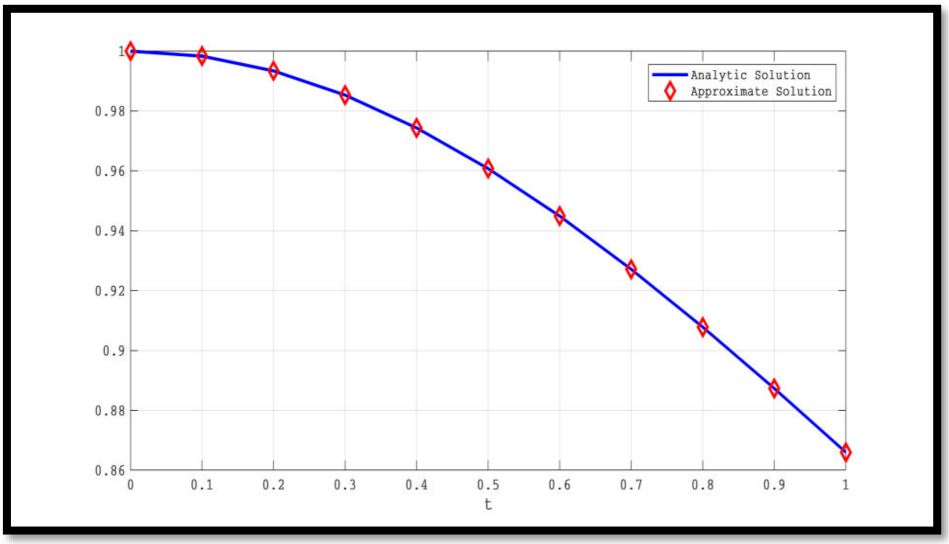


Figure 2. Exact and approximate solution for example 1

Example 2. Consider the Emden–Fowler equation of the second kind (Singh and Kumar 2014b; Singh, Kumar, and Nelakanti 2013)

$$(tu'(t))' = -te^{u(t)},\ 0 < t < 1$$

$$u'(0) = 0,\ u(1) = 0 \tag{27}$$

This problem is also a particular of Eq. (1) - Eq. (3). That means

$\alpha = 1,\ a_1 = 1,\ a_2 = 0,\ b = 0$ and $f(t, u(t)) = -e^{u(t)}$

The exact solution of Eq. (27) is

$$u(t) = \ln\left(\frac{\omega + 1}{\omega t^2 + 1}\right)^2, \quad \omega = 3 \pm 2\sqrt{2}.$$

The approximate solution using the proposed method and the absolute errors from the exact solution for this problem are tabulated in Table 2 and Figure 3 shows the graphs of exact and approximate solutions. The maximum absolute error using n=10 in (Singh and Kumar 2014b) is 2.1007×10^{-6} but in the proposed method the maximum absolute error is 6.08010002×10^{-7}.

Table 2: Quintic B-spline results and absolute errors for Example 2

t	Proposed Method	Exact	Absolute Error
0.0	0.3166943136905000	0.3166943676407495	0.0000000539502495
0.1	0.3132658116002000	0.3132658504980633	0.0000000388978633
0.2	0.3030154836333000	0.3030154228322998	0.0000000608010002
0.3	0.2860472317263000	0.2860472653048539	0.0000000335785539
0.4	0.2625311115885000	0.2625311274560331	0.0000000158675331
0.5	0.2326967637107000	0.2326967838738344	0.0000000201631344
0.6	0.1968268234783000	0.1968268056929538	0.0000000177853462
0.7	0.1552481356735900	0.1552481066827563	0.0000000289908337

0.8	0.1083227083331800	0.1083227634444646	0.0000000551112846
0.9	0.0564386382555000	0.0564386024692362	0.0000000357862638
1.0	0.0000000000000000	0.0000000000000000	0.0000000000000000

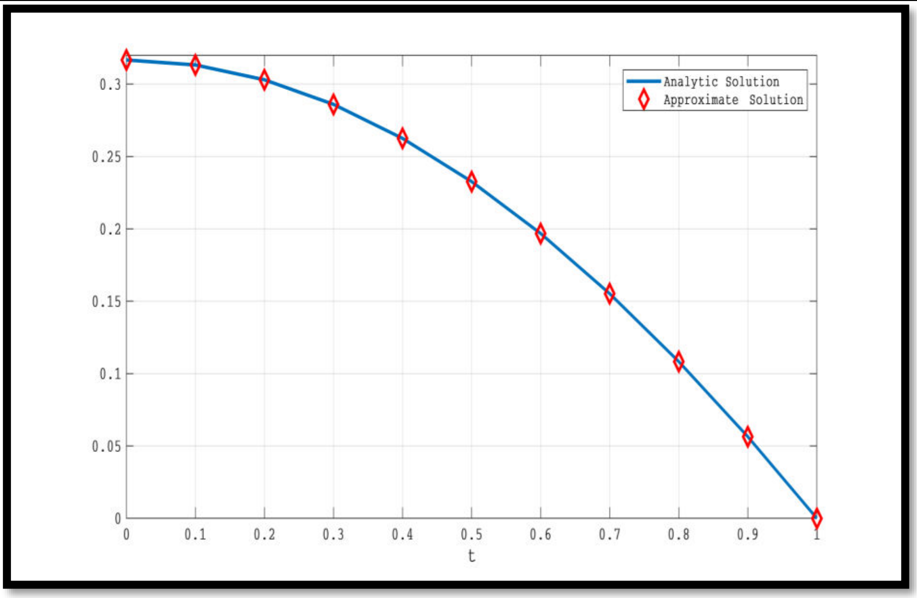


Figure 3. Exact and approximate solution for example 2

Example 3. Consider the problem arising in steady-state oxygen diffusion in a spherical cell (Singh and Kumar 2014b)

$$(t^2 u'(t))' = t^2 \frac{0.76129\, u(t)}{u(t)+0.03119},\ \ 0 < t \le 1$$

$$u'(0)=0,\ \ 5u(1)+u'(1)=5 \tag{28}$$

This problem is also a particular of Eq. (1) - Eq. (3). That means

$$\alpha = 2,\ a_1 = 5,\ a_2 = 1,\ b = 5 \text{ and } f(t,u(t)) = \frac{0.76129\, u(t)}{u(t)+0.03119}$$

Since there is no exact solution to this problem, Table 3 lists the absolute residual error along with the approximate solution using the proposed method. The table demonstrates how well the approximation fits the given differential equation as compared with (Singh and Kumar 2014b) and Figure 4 shows the graph of approximate solution.

Table 3: Quintic B-spline results and absolute residual errors for Example 3

t	Proposed Method	Absolute Residual Error
0.0	0.8284832738671110	.0000000000048907
0.1	0.8297060896199600	0.0000000000130954
0.2	0.8333749932953860	0.0000000000013221
0.3	0.8394898608255860	0.0000000000144343
0.4	0.8480527132514780	0.0000000000027344
0.5	0.8590648306050520	0.0000000000191241
0.6	0.8725281920852540	0.0000000000056047

0.7	0.8884451398822570	0.0000000000181770
0.8	0.9068183380793901	0.0000000000038411
0.9	0.9276507281761430	0.0000000000165969
1.0	0.9509456827908310	0.0000000000014442

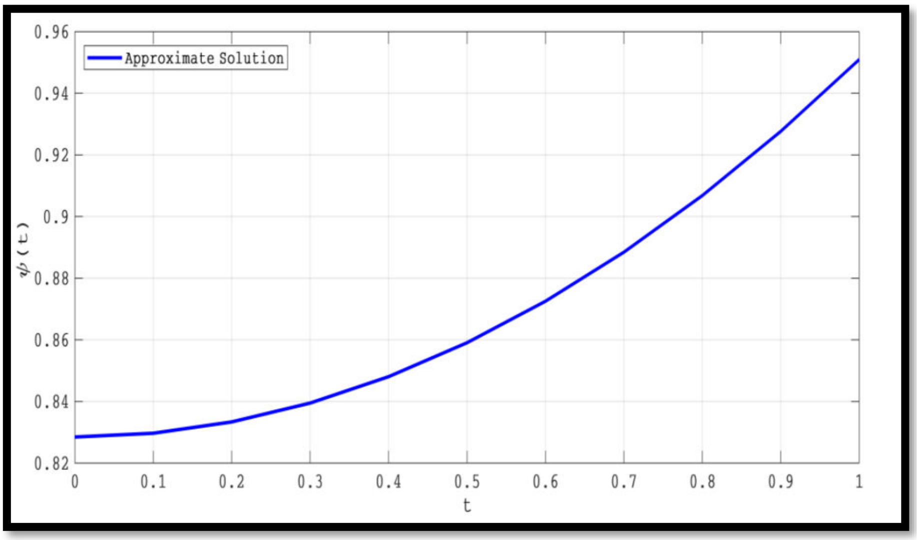


Figure 4. Approximate solution for example 3

Example 4. Consider the problem arising in the distribution of heat sources in the human head (Duggan and Goodman 1986)

$$(t^2 u'(t))' = -t^2 e^{-u(t)}, \; 0 < t \le 1$$

$$u'(0) = 0, \; 2u(1) + u'(1) = 0 \tag{29}$$

This problem is also a particular of Eq. (1) - Eq. (3). That means

$$\alpha = 2, \; a_1 = 2, \; a_2 = 1, \; b = 0 \text{ and } f(t, u(t)) = -e^{-u(t)}$$

Since there is no exact solution to this problem, Table 4 lists the absolute residual error along with the approximate solution using the proposed method. The table demonstrates how well the approximation fits the given differential equation as compared with (Singh and Kumar 2014b) and Figure 5 shows the graph of approximate solutions.

Table 4: Quintic B-spline results and absolute residual errors for Example 4

t	Proposed Method	Absolute Residual Error
0.0	0.2707362706946570	0.0000076472474255
0.1	0.2694577343898710	0.0000087187338805
0.2	0.2656153095373430	0.0000004402308103
0.3	0.2591936707452740	0.0000089784525717
0.4	0.2501643551645480	0.0000001791309878
0.5	0.2384857808655290	0.0000028924677682
0.6	0.2241052734435516	0.0000089889558170
0.7	0.2069564526020600	0.0000098918597821
0.8	0.1869558637840680	0.0000027894342962
0.9	0.1640051347248120	0.0000019795197907

1.0	0.1379877451258660	0.0000008236639641

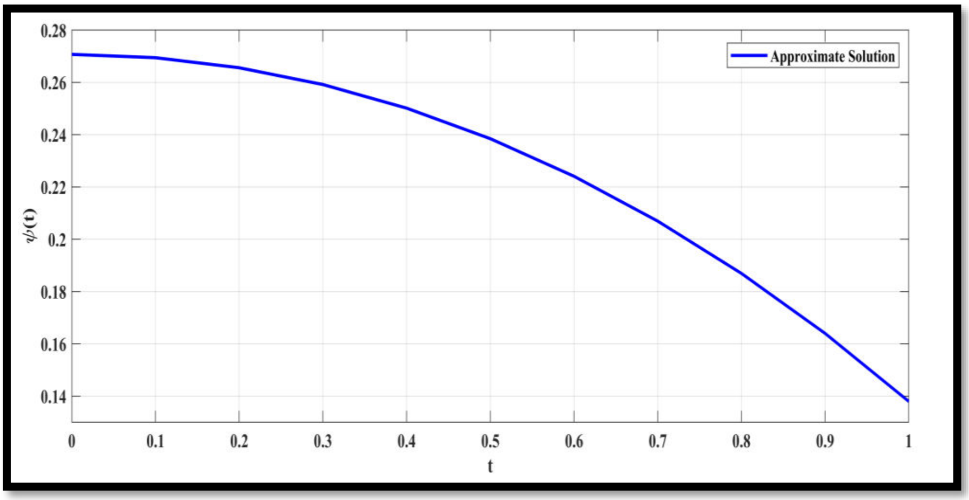


Figure 5. Approximate solution for example 4

6. Conclusion

This paper introduces the quintic B-spline collocation method in order to approximate the solution of a class of non-linear singular boundary value problems with Neumann and Robin boundary conditions that arise in various models. Using L'Hopital's rule, we first modify the problem at the singular point in order to solve the given class of second order differential equations. Then, we transform the problem to apply the proposed method. With quintic B-spline approaches systems of nonlinear equations are generated, and these systems can be handled with appropriate methods with little difficulty in terms of computation and time. The method yields uniform convergence; as demonstrated by the numerical results. Furthermore, the spline function that is generated by this method can be utilized to find the solution at any point in the range. Four widely applicable problems in different models are solved using the proposed approach in order to test the method's accuracy. The proposed method produced numerical results in Example 1 and Example 2 that are better than (Singh and Kumar 2014b) in terms of maximum absolute error and in good agreement with the exact solutions found in the literature. The proposed approach yields better absolute residual error in Example 3 and Example 4 that their solutions are not known than (Singh and Kumar 2014b), demonstrating how well the approximation satisfies the given non-linear singular boundary value problems. The proposed method overcomes the major limitation of existing numerical methods in the literature, which is that they require a significant amount of computational cost for nonlinear singular boundary value problems by calculating unknown constants in a series of difficult transcendental equations and extracting extra terms from the approximate solution (Singh and Kumar 2014b; Singh, Kumar, and Nelakanti 2013). In addition to handling singularities, the proposed method manages larger domains and offers solutions for problems with strong non-linearity.

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