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Optimizing Inventory Model for a Time Varying Degradation and Linear Holding Cost with Split Demand Under Non-Decreasing Shortages

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Abstract

This study presents a mathematical model for analyzing inventory systems with time-dependent deteriorating items and holding costs vary linearly over time. The model accommodates expected shortages with a backlog rate that evolves. Demand is characterized by two distinct functions: a time-dependent quadratic function during periods without shortages and a time-dependent linear function during periods with shortages. Computational Algorithm is formulated to obtain the minimal total cost, and the convexity of the full cost function is established. Additionally provide a numerical example to demonstrate the application of the model. Furthermore, sensitivity analysis of optimal inventory policies is conducted, and the impact of decision variable variations are graphically represented using MATLAB. This comprehensive framework contributes to a deeper understanding of inventory management under fluctuating demand conditions.

Keywords: EOQ; Quadratic demand; Linear demand; Shortage; time varying deteroration; linear holding cost; Computational Algorithm;

1 Introduction

The working capital of any business can be effectively optimized through inventory management, making it a vital aspect of a product-based business. The products in an inventory encompass physical resources, unprocessed materials, and finished commodities. The primary objective of any inventory system is to devise strategies and policies that minimize the overall cost of holding the stock. Information technology plays a pivotal role in aiding decision-making through the analysis of optimal ordering or manufacturing policies. Inventory management has its share of difficulties, including product damage and obsolescence, leading to shortages.

2. Literature review

The model developed in this study is based on four lines of inventory theory, which has its roots in Harris's (1915) classical economic quantity model. These streams include linear holding costs, time-dependent split demand, deteriorating items with expiration dates, and item shortages within the inventory cycle.

2.1 Inventory models for degradation items

Deterioration is a natural phenomenon in inventory systems and holds significant importance. Food products deteriorate due to spoilage or unsafe storage conditions, while fuels like petrol, alcohol, and camphor degrade through evaporation. Technology and fashion relate to obsolescence, which can be considered deterioration and cause a decline in inventory value, especially in clothing and electronic products. Several studies have explored the impact of stock deterioration on profitability using different deterioration rates. Whitin (1957) was the pioneer in research on inventory degradation, focusing on fashion items. Ghare and Schrader (1963) used an exponential decay rate, while Covert and Philip (1973) introduced a two-parameter Weibull-distributed deterioration. Donaldson (1977) examined the conventional no-shortage inventory model for depreciating goods with a linear demand trend.

Some of the recent studies on inventories with deterioration include Sarkar and Sarkar's (2013) survey on time-varying deterioration under stock-associated demand, Khan et al. (2019-2020) study on variable deterioration rates based on storage length and product lifespan, Abu Hashan's (2020) inventory model for deteriorating items with different type of demands, Jani et al. (2021) model on food items in India. Shaikh et al.(2021) EOQ system for deteriorating items with credit policies; in 2022, Al-Amin Khan et al. proposed inventory management strategies with hybrid cash-advance payments for time-dependent demand and deteriorating items.

2.2 Inventory models for time varying holding cost

The holding cost was assumed to be predictable and constant in conventional inventory models. In reality, this is not always true in the case of storage of products that are deteriorating and perishable, such as foodstuffs, milk, fruit, vegetables, and meat, whose quality declines with each passing day and rising holding expenses; hence, it necessitates preserving the items and avoiding spoilage. Additionally, because of factors like inflation, bank interest, hiring fees, etc., that rise with time, certain factors that influence the holding cost remain constant while others change. Hence, Assuming that the holding cost per unit of time changes with time is logical.

Ali Akbar Shaikh et al. (2019) explored price discount facilities in an EOQ model for deteriorating items with partial backlogging. An inventory model with incremental holding cost under partial backlogging was developed by Singh and Sharma (2019). Khan et al. investigated an inventory system with an expiration date, pricing, and replenishment decisions in 2019 and then discussed advanced payment and linearly time-dependent holding costs. Their study was also extended to demand dependent on advertisement and selling prices in 2020. Sivashankari and Vijayakumar (2023) recently discussed stock-dependent demand effects in EOQ models with various holding costs.

2.3 Inventory models for shortages

Another crucial element of the inventory model is the stock-out scenario. When there is still a market for a product, the shop periodically runs out of it. The term "inventory backlog" refers to the delayed delivery of items to clients. Customers may decide to wait for the delayed delivery or go somewhere else in consideration of the retailer's goodwill. The retailer still prepares the stock for the waiting customers despite the loss at the next store. As a result, inventory backlog has a negative side in that it negatively affects inventory goodwill.

Shaikh et al. (2018) analyzed an economic order quantity model for degrading commodities with trade credit, partial backlog, preservation technologies, and financial factors. Further analysis of an EOQ model with backlogging was carried out by Ghandehari and Dezhtaherian (2019). In 2021, Saurabh Srivastava and Rajesh Kumar Bajaj explored an optimal inventory management system in a fuzzy setup. Senbagam and Kokilamani (2023) proposed inventory models with partial back ordering in an imprecise environment.

2.4 Inventory models for time dependent demand

Numerous topics are covered in these studies, including demand patterns, ramp, time, selling price, and combinations of time and demand, as well as linear and quadratic demand functions. The cost and quantity of a product directly affect its level of popularity. Customer demand declines as prices rise, lengthening the inventory cycle. Hence, maintaining a sizable inventory of goods will increase the number of customers. It is important to note that this impacts profit margins because significant investment is required. An increase in inventory also affects the inventory holding cost, including maintenance and spoilage. Sometimes, depending on the socio-economic standing of the local populace, demand may be reduced. This harms inventory.

Ghosh and Chaudhuri's (2004) inventory model includes continuous deterioration, time-dependent quadratic demand, and shortages in all cycles. Yang 2005 discussed the fluctuation of time-dependent demand on degrading goods inventory. Probabilistic demand has gained attention in the corporate environment due to uncertainty. Models have been developed to help sellers choose optimal ordering tactics and delay spending for shortages. Examples include Khanra et al. (2011) model for failing products with time-linked demand, Jaggi et al. (2013) imperfect EOQ model, Yadav and Vat's (2014) spoiling goods inventory model, Sharmila and Uthayakumar's (2015) EOQ model for quadratic demand and partial backlogging, Ali Akbar Shaikh et al.'s (2018) preservation technology model. Tripathi (2018) Time liked the Quadratic demand model with salvage values.

Kumar presented a model for linearly rising deterministic demand and quadratically increasing holding costs in 2019. Further, Tripathi (2019) investigated a time-linked demand and shortage inventory model for failed goods. Setiawan presented a two-parameter Weibull degradation inventory model with quadratic time-sensitive demand and linearly time-dependent shortages in 2021. Many researchers, Rahaman (2020), Balarama (2020), Garima (2022), Babangida (2022), etc., have all developed Quadratic time-linked demand with various patterns of degradations with or without shortages. Priti Chaudhary and Tanuj Kumar (2022) developed an Intuitionistic fuzzy inventory model for a quadratic demand rate. Log-gamma-deteriorating items with quadratic demand and shortages were investigated by Senbagam and Kokilamani in 2022.

Rukonuzzaman et al. (2023) studied mango businesses in Bangladesh to explore quantity-based discount frames.

The demand rate for most products fluctuates with time in a patterned manner, more often represented as a function; it may be dependent on stock, time, selling price, and advertisement. In this paper, the demand rate is considered time-dependent and split. Split patterns of demand are common in the market's newly introduced products. This demand frequently occurs due to customer perceptions of products and external factors like advertisements. In this context, any trade agreement will result in higher order quantities and overall costs if initial demand rises.

The finest example of fluctuating demand is the split and time-linked demand, represented as Phase 1: $D(t) = a+bt+ct^2$, where $a > 0$, $b \neq 0$, and $c \neq 0$. The demand rate is a rising function of time if $b > 0$ and $c > 0$. If $b > 0$ and $c < 0$, the demand rate slows down. This is referred to as an accelerated expansion in demand. If $b < 0$ and $c < 0$, then the demand rate will drop slowly each time. Phase 2: Demand turns out to be linear.

2.5 Research questions

Based on the review outlined above, a list of important research issues has been compiled

1. How do you cope with periodic fluctuations in demand?
2. To minimize the retailer's expenses and determine the Economic Order Quantity.
3. How is this model used in a real-world business setting?
4. How do you manage the stock-out period of the inventory?
5. What are the effects of the consumption rate that relies on storage duration at the company's ideal inventory?
6. How does the instant degradation function impact the company's inventory planning?

Earlier research utilized the demand function over the entire inventory, but this research incorporates two different forms of demand, as mentioned above, without shortage and with shortage, respectively.

The primary assumptions used to develop the innovative inventory model are:

(i) Two levels of time-varying demand

(ii) Partial backlogging is based on the length of the customer’s waiting time.

(iii) Time-dependent deterioration.

(iv) The holding cost is a linearly time-dependent increasing function.

Inventory policies are derived for the proposed inventory model to minimize the total cost. A numerical example illustrates the solution procedure of the proposed optimization model. Also, a sensitivity analysis with the effects of system constraints is discussed.

3 Formulation of the problem

3.1 Assumptions

- The inventory consists of a single item only.
- The cost to hold a unit product for a unit time duration is proportional to the duration of the storage time of that unit: $\wp = \mu + \nu t$, $\mu \geq 0$, and $\nu \geq 0$.
- The degradation rate $\theta(t)$ is time-linked; $\theta(t) = \theta t$, $0 \leq \theta < 1$.
- The planning horizon is infinite.

- Demand rate $\xi(t)$ is time-dependent and follows a split pattern during a cycle. $\xi_1(t) = \alpha+\beta t+\gamma t^2$, $0 \le t < t_1$ and $\xi_2(t) = \alpha+\beta t$, $t_1 < t \le T$.
- The replenishment rate is instantaneous, and the item is replenished periodically (each inventory cycle).
- Deficiencies are allowed and fully backlogged. The backlog rate is a linear function of time.
- Lead time is zero.

3.2 Notations

$\chi(t)$ - Inventory level at instant 't'
A - Cost of placing an order
λ - The unit cost of a deteriorating item
η - Shortage cost for backlogged items per unit per cycle
t_1 - Duration of physical stock entering the storage, shortages occurring just after t_1
T - Length of cycle time or scheduled period
Ω - Ordering cost
Υ - Backorder cost
Ψ - Holding cost
Φ - Degradation cost
Q - (Q_1+Q_2) the order size per cycle
$\Pi(T,t_1)$ - The total cost of the inventory system

4 Mathematical Formulation

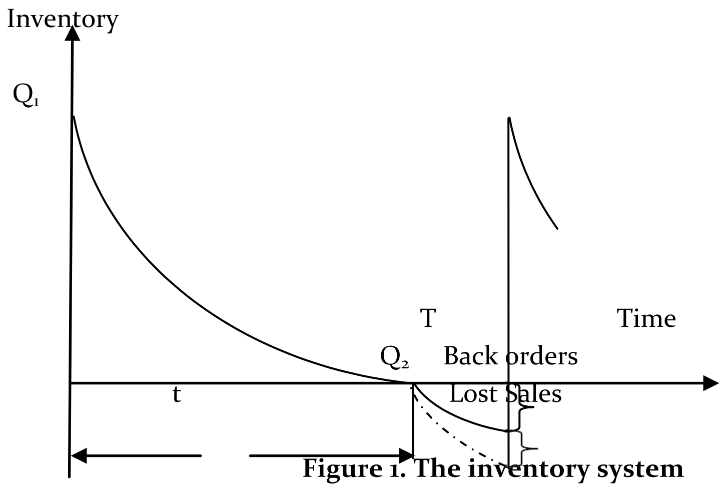


Figure 1. The inventory system

In the beginning, the inventory was solely attributable to customer consumption. However, the stock is reduced not only to meet customer demand but also because of degradation, and as a result, the inventory amount reaches zero at time t_1. This situation can be modeled as a differential equation.

$$\frac{d\chi_1(t)}{dt} + \theta(t)\chi_1(t) = -\xi_1(t),\ 0 \le t < t_1 \qquad (1)$$

with the boundary condition $\chi_1(t_1) = 0$ at $t = t_1$.

During the interval $[t_1, T]$, a stock-out situation arises. Inventory during $[t_1, T]$ can be represented by the differential equation.

$$\frac{d\chi_2(t)}{dt} = -\xi_2(t),\ t_1 < t \le T \tag{2}$$

Solution of (1) and (2) are

$$\chi_1(t) = \left(\alpha t_1 + \frac{\beta t_1^2}{2} + \frac{\gamma t_1^3}{3}\right) - \left(\alpha t + \frac{\beta t^2}{2} + \frac{\gamma t^3}{3}\right)$$
$$+\theta\left\{\left(\frac{\alpha t_1^3}{6} + \frac{\beta t_1^4}{8} + \frac{\gamma t_1^5}{10}\right) - \left(\frac{\alpha t_1 t^2}{2} + \frac{\beta t_1^2 t^2}{4} + \frac{\gamma t_1^3 t^2}{6}\right) + \left(\frac{\alpha t^3}{3} + \frac{\beta t^4}{8} + \frac{\gamma t^5}{15}\right)\right\}$$
$$+\theta^2\left\{\left(\frac{\alpha t_1^5}{40} + \frac{\beta t_1^6}{48} + \frac{\gamma t_1^7}{56}\right) - \left(\frac{\alpha t_1^3 t^2}{12} + \frac{\beta t_1^4 t^2}{16} + \frac{\gamma t_1^5 t^2}{20}\right) + \left(\frac{7\alpha t^5}{120} + \frac{\beta t^6}{24} + \frac{9\gamma t^7}{280}\right)\right\} \tag{3}$$

$$\chi_2(t) = \alpha(t - t_1) + \frac{\beta}{2}\left(t^2 - t_1^2\right) \tag{4}$$

with $Q_1 = \chi_1(0)$ and $Q_2 = \chi_2(T)$, $Q = Q_1 + Q_2$

$$Q = \left(\alpha t_1 + \frac{\beta t_1^2}{2} + \frac{\gamma t_1^3}{3}\right) + \theta\left(\frac{\alpha t_1^3}{6} + \frac{\beta t_1^4}{8} + \frac{\gamma t_1^5}{10}\right) + \theta^2\left(\frac{\alpha t_1^5}{40} + \frac{\beta t_1^6}{48} + \frac{\gamma t_1^7}{56}\right) + (T - t_1)\left(\alpha + \frac{\beta}{2}(T + t_1)\right) \tag{5}$$

4.1 Various cost functions of this model

(i) Retailers handle various responsibilities like creating purchase orders, shipping, inspecting, processing, storing, and reporting products, considering the total acquisition cost. The cost of ordering is Ω.

With the values of $\chi_1(t)$ and $\chi_2(t)$, deterioration cost, holding cost and shortage costs are found.

(ii) Number of deteriorating items during $[0, t_1]$ is

$$Q_1 - \int_0^{t_1}\left(\alpha + \beta t + \gamma t^2\right)dt = \theta\left(\frac{\alpha t_1^3}{6} + \frac{\beta t_1^4}{8} + \frac{\gamma t_1^5}{10}\right) + \theta^2\left(\frac{\alpha t_1^5}{40} + \frac{\beta t_1^6}{48} + \frac{\gamma t_1^7}{56}\right) \tag{6}$$

(iii) The loss resulting from deterioration, damage, and obsolescence is called the "cost of deterioration". It decreases with time between $[0, t_1]$, with the initial stock still at the conclusion. The cost of deteriorating items is

$$\Phi = \lambda\left\{Q_1 - \int_0^{t_1}\left(\alpha + \beta t + \gamma t^2\right)dt\right\} = \lambda\theta\left\{\left(\frac{\alpha t_1^3}{6} + \frac{\beta t_1^4}{8} + \frac{\gamma t_1^5}{10}\right) + \theta\left(\frac{\alpha t_1^5}{40} + \frac{\beta t_1^6}{48} + \frac{\gamma t_1^7}{56}\right)\right\} \tag{7}$$

(iv) Retailers have an adequate supply of products on hand to satisfy customer demand, and technology saves costs by providing access to store inventory at various times. The cost of holding items is

$$\Psi = \int_0^{t_1}(\mu + \nu t)\chi_1(t)dt = \mu\left[\left(\frac{\alpha t_1^2}{2} + \frac{\beta t_1^3}{3} + \frac{\gamma t_1^4}{4}\right) + \theta\left(\frac{\alpha t_1^4}{12} + \frac{\beta t_1^5}{15} + \frac{\gamma t_1^6}{18}\right) + \theta^2\left(\frac{\alpha t_1^6}{144} + \frac{\beta t_1^7}{168} + \frac{\gamma t_1^8}{192}\right)\right]$$
$$+\nu\left[\left(\frac{\alpha t_1^3}{6} + \frac{\beta t_1^4}{8} + \frac{\gamma t_1^5}{10}\right) + \theta\left(\frac{\alpha t_1^5}{40} + \frac{\beta t_1^6}{48} + \frac{\gamma t_1^7}{56}\right)\right] \tag{8}$$

(v) Shortage costs result from unfulfilled demand and are mathematically represented as Υ.

$$\Upsilon = \eta\int_{t_1}^{T}\left(-\chi_2(t)\right)dt = \frac{\eta\left(T-t_1\right)^2}{6}\left(3\alpha+\beta\left(T+2t_1\right)\right) \tag{9}$$

Thus, the retailer's total cost per cycle is $\Pi(T, t_1) = (\Omega + \Phi + \Psi + \Upsilon)/T$

$$\Pi(T,t_1) = \frac{1}{T}\left\{A+\mu\left[\left(\frac{\alpha t_1^2}{2}+\frac{\beta t_1^3}{3}+\frac{\gamma t_1^4}{4}\right)+\theta\left(\frac{\alpha t_1^4}{12}+\frac{\beta t_1^5}{15}+\frac{\gamma t_1^6}{18}\right)+\theta^2\left(\frac{\alpha t_1^6}{144}+\frac{\beta t_1^7}{168}+\frac{\gamma t_1^8}{192}\right)\right]\right.$$
$$\left.+\left(\nu+\lambda\theta\right)\left[\left(\frac{\alpha t_1^3}{6}+\frac{\beta t_1^4}{8}+\frac{\gamma t_1^5}{10}\right)+\theta\left(\frac{\alpha t_1^5}{40}+\frac{\beta t_1^6}{48}+\frac{\gamma t_1^7}{56}\right)\right]+\frac{\eta\left(T-t_1\right)^2}{6}\left(3\alpha+\beta\left(T+2t_1\right)\right)\right\} \tag{10}$$

which is a non-linear function. Hence, the non-linear optimization problem can be formulated as

$$\text{Minimize } \Pi = \frac{X}{T}. \text{ Subject to } 0 \le t_1 \le T \tag{11}$$

where
$$X = \left\{A+\mu\left[\left(\frac{\alpha t_1^2}{2}+\frac{\beta t_1^3}{3}+\frac{\gamma t_1^4}{4}\right)+\theta\left(\frac{\alpha t_1^4}{12}+\frac{\beta t_1^5}{15}+\frac{\gamma t_1^6}{18}\right)+\theta^2\left(\frac{\alpha t_1^6}{144}+\frac{\beta t_1^7}{168}+\frac{\gamma t_1^8}{192}\right)\right]\right.$$
$$\left.+\left(\nu+\lambda\theta\right)\left[\left(\frac{\alpha t_1^3}{6}+\frac{\beta t_1^4}{8}+\frac{\gamma t_1^5}{10}\right)+\theta\left(\frac{\alpha t_1^5}{40}+\frac{\beta t_1^6}{48}+\frac{\gamma t_1^7}{56}\right)\right]+\frac{\eta\left(T-t_1\right)^2}{6}\left(3\alpha+\beta\left(T+2t_1\right)\right)\right\}$$

4.2 Computational steps to obtain the optimal Solution

Step 1: Define the parameters to α, β, γ, A, μ, ν, λ, η and θ.
Step 2: Determine the holding, deterioration, and backorder cost. The entire cost of the cycle can be obtained by substituting all cost values in equation (10).
Step 3: Compute the first-order partial derivatives, $\frac{\partial\Pi(T,t_1)}{\partial t_1}$ and $\frac{\partial\Pi(T,t_1)}{\partial T}$ for equation (10).
Step 4: Set $\frac{\partial\Pi(T,t_1)}{\partial t_1}=0$ and $\frac{\partial\Pi(T,t_1)}{\partial T}=0$, to obtain the extremum of Π.
Step 5: Ensure the optimality conditions

$$\frac{\partial^2\Pi(T,t_1)}{\partial T^2}>0, \frac{\partial^2\Pi(T,t_1)}{\partial t_1^2}>0 \text{ and } \left(\frac{\partial^2\Pi(T,t_1)}{\partial T^2}\right)\left(\frac{\partial^2\Pi(T,t_1)}{\partial t_1^2}\right)-\left(\frac{\partial^2\Pi(T,t_1)}{\partial T\partial t_1}\right)^2>0$$

Step 6: The Hessian matrix of second derivatives

$$H(T,t_1)=\begin{vmatrix}\frac{\partial^2X(T,t_1)}{\partial T^2} & \frac{\partial^2X(T,t_1)}{\partial t_1\partial T}\\ \frac{\partial^2X(T,t_1)}{\partial T\partial t_1} & \frac{\partial^2X(T,t_1)}{\partial t_1^2}\end{vmatrix}$$

Determining the convexity of the function $X(T, t_1)$ establishes the concavity of the nonlinear function (10). The optimal values for the retailer's total cost and economic order quantity are also computed.

4.3 Hypothetical outcome

This section establishes the hypothetical outcome of a function of the form $\Pi(x)=\dfrac{f(x)}{g(x)}$.

Theorem 1

The overall minimum value of the total cost function $\Pi(T, t_1)$ occurs at the point (t_1^*, T^*) if the total cost function $\Pi(T, t_1)$ is a strictly pseudo-convex function in t_1 and T.

Proof:

From the retailer's total cost function (10), for convenience, let us take the following auxiliary functions:

$$f(T,t_1)=\left\{A+\mu\left[\left(\frac{\alpha t_1^2}{2}+\frac{\beta t_1^3}{3}+\frac{\gamma t_1^4}{4}\right)+\theta\left(\frac{\alpha t_1^4}{12}+\frac{\beta t_1^5}{15}+\frac{\gamma t_1^6}{18}\right)+\theta^2\left(\frac{\alpha t_1^6}{144}+\frac{\beta t_1^7}{168}+\frac{\gamma t_1^8}{192}\right)\right]\right.$$
$$\left.+(\nu+\lambda\theta)\left[\left(\frac{\alpha t_1^3}{6}+\frac{\beta t_1^4}{8}+\frac{\gamma t_1^5}{10}\right)+\theta\left(\frac{\alpha t_1^5}{40}+\frac{\beta t_1^6}{48}+\frac{\gamma t_1^7}{56}\right)\right]+\frac{\eta(T-t_1)^2}{6}\big(3\alpha+\beta(T+2t_1)\big)\right\}$$

and $g(T, t_1) = T > 0$.

The task requires proving that $f(T, t_1)$ is a non-negative, differentiable, and strictly joint convex function. The second-order partial derivatives of $f(T, t_1)$ about t_1 and T is determined are

$$\frac{\partial^2 f}{\partial T^2}=b(\alpha+\beta T)>0$$

$$\frac{\partial^2 f}{\partial t_1\partial T}=-b(\alpha+\beta t_1)$$

$$\frac{\partial^2 f}{\partial t_1^2}=\left\{c_1\left[\left(\alpha+2\beta t_1+3\gamma t_1^2\right)+\theta\left(\alpha t_1^2+\frac{4\beta t_1^3}{3}+\frac{5\gamma t_1^4}{3}\right)+\theta^2\left(\frac{5\alpha t_1^4}{24}+\frac{\beta t_1^5}{4}+\frac{7\gamma t_1^6}{24}\right)\right]\right.$$
$$\left.+(c_2+p\theta)\left[\left(\alpha t_1+\frac{3\beta t_1^2}{2}+2\gamma t_1^3\right)+\theta\left(\frac{\alpha t_1^3}{2}+\frac{5\beta t_1^4}{8}+\frac{3\gamma t_1^5}{4}\right)\right]+b\big(\alpha-\beta(T-2t_1)\big)\right\}$$

Hence, the Hessian matrix for $f(T, t_1)$ is

$$H(T,t_1)=\begin{vmatrix}\dfrac{\partial^2 f(T,t_1)}{\partial T^2} & \dfrac{\partial^2 f(T,t_1)}{\partial t_1\partial T}\\ \dfrac{\partial^2 f(T,t_1)}{\partial T\partial t_1} & \dfrac{\partial^2 f(T,t_1)}{\partial t_1^2}\end{vmatrix}$$

The first principal minor $|H_{11}|$ is a more significant zero. Also, the second principal minor $|H_{22}|=\left(\dfrac{\partial^2 f}{\partial T^2}\right)\left(\dfrac{\partial^2 f}{\partial t_1^2}\right)-\left(\dfrac{\partial^2 f}{\partial t_1\partial T}\right)^2$ is more effective than zero.

The Hessian matrix for $f(T, t_1)$ is positive-definite, making it a non-negative, differentiable, and convex function. The total cost function per unit time, $\Pi(T, t_1)$, is a pseudo-convex function with only one minimum value due to the positive, differentiable concave function $g(T, t_1)$. As a result, at the point (T^*, t_1^*), the objective function $\Pi(T, t_1)$ achieves its minimum.

In the presented inventory model, the total cost function exhibits high nonlinearity, making it challenging to directly find the optimum values of decision variables. Therefore, the computational algorithm outlined in the study is employed to obtain the optimal solution by substituting numerical values for all parameters in the total cost function, excluding decision variables. This approach allows for practical results that reflect real-world business environments.

By following the computational algorithm, the optimal solution can be derived iteratively, considering the numerical values of demand, ordering cost, backlog parameter, and others. This iterative process enables retailers to determine the most cost-effective inventory management strategies based on the specific characteristics of their business operations.

Ultimately, by utilizing numerical values in the total cost function and following the computational algorithm, retailers can obtain practical and actionable insights into optimizing inventory decisions, enhancing overall efficiency, and maximizing profitability in dynamic business environments.

5 Numerical illustration and Sensitivity Analysis

To illustrate and validate the proposed model to represent reality, appropriate numerical data is considered, and the optimal values are found. A sensitivity analysis of all input parameters are carried out.

5.1 Numerical Example

Consider the values α = 2000 units per year, β = 200 units per year, γ = 20 units per year, A = \$200 per order, μ = \$2 per year, ν = \$0.05 per year, λ = \$20 per unit, η = \$5 per year and θ = 0.01. The optimal values obtained for the above data for the total cost function (10), is determined as Π(Total cost) = \$1084.50, t_1 = 0.2585, T = 0.3637 and Q = 741units per cycle in appropriate units.

5.2 Sensitivity Analysis

Table 1. Sensitivity analysis for all input parameters

Parameter	Changes in parameter Value	t_1	T	Q	$\Pi(\$)$
α	1600	0.2864	0.4032	661.6407	974.34
	1800	0.2714	0.3819	702.1892	1030.53
	2000	0.2585	0.3637	740.8846	1084.51
	2200	0.2473	0.3478	777.4129	1135.69
	2400	0.2374	0.3339	812.5609	1184.72
β	100	0.2611	0.3637	741.5671	1079.01
	150	0.2598	0.3655	741.2244	1081.77
	200	0.2585	0.3637	740.8846	1084.51
	250	0.2573	0.3620	740.4661	1087.21
	300	0.2561	0.3603	740.1553	1089.89

γ	16	0.2586	0.3638	740.9254	1084.48
	18	0.2586	0.3637	740.8901	1084.49
	20	0.2585	0.3637	740.8846	1084.51
	22	0.2585	0.3637	740.8405	1084.52
	24	0.2585	0.3637	740.7954	1084.53
A	160	0.2320	0.3262	663.2425	968.57
	180	0.2457	0.3456	703.2108	1028.11
	200	0.2585	0.3637	740.8846	1084.51
	220	0.2707	0.3809	776.5187	1138.22
	240	0.2823	0.3974	810.8642	1189.61
μ	1.8	0.2756	0.3768	768.0866	1045.31
	1.9	0.2668	0.3701	753.9879	1065.37
	2.0	0.2585	0.3637	740.8846	1084.51
	2.1	0.2509	0.3579	728.7705	1102.80
	2.2	0.2437	0.3525	717.5234	1120.30
ν	0.03	0.2588	0. 3639	741.2698	1084.18
	0.04	0.2587	0.3638	741.1113	1084.35
	0.05	0.2585	0.3637	740.8846	1084.51
	0.06	0.2584	0.3636	740.6633	1084.67
	0.07	0.2583	0.3635	740.4509	1084.83
λ	10	0.2598	0.3648	742.9931	1082.88
	15	0.2591	0.3643	741.9717	1083.70
	20	0.2585	0.3637	740.8846	1084.51
	25	0.2579	0.3632	739.7166	1085.31
	30	0.2574	0.3627	738.7454	1086.11
η	3	0.2370	0.3875	810.8827	993.23
	4	02498	0.3767	767.7577	1047.39
	5	0.2585	0.3637	740.8846	1084.51
	6	0.2649	0.3548	722.3449	1111.59
	7	0.2698	0.3482	708.8014	1132.23
θ	0.005	0.2598	0.3648	742.9633	1082.86
	0.010	0.2585	0.3637	740.8846	1084.51
	0.015	0.2573	0.3627	738.7503	1084.13
	0.020	0.2561	0.3616	736.5673	1087.74
	0.025	0.2550	0.3607	734.5800	1089.33

5.3 Graphical Representation of the Total cost with various parameters

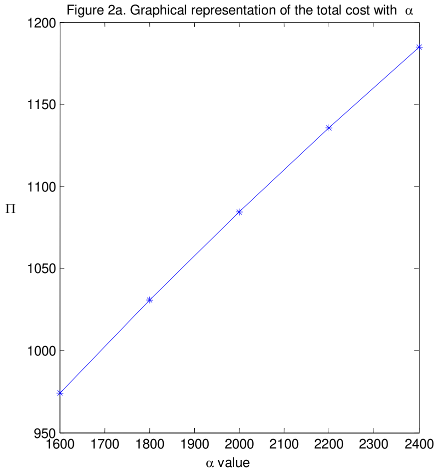


Figure 2a. Graphical representation of the total cost with α

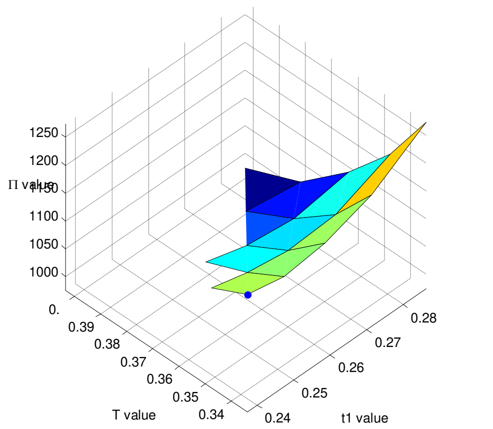


Figure 2b. Graphical representation of the total cost with t1 and T

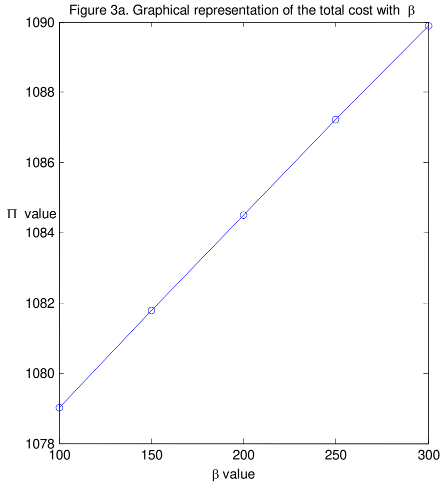


Figure 3a. Graphical representation of the total cost with β

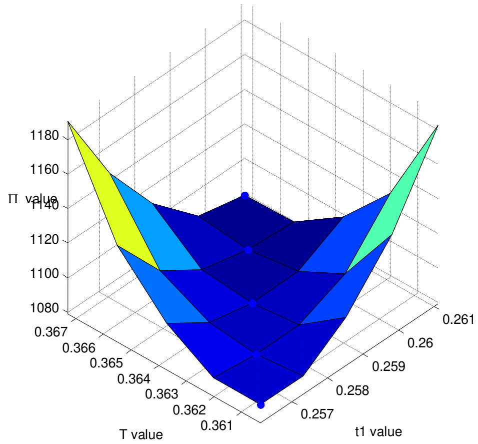


Figure 3b. Graphical representation of the total cost with t1 and T

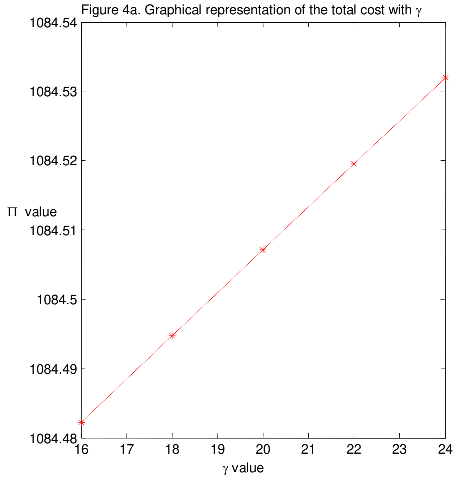


Figure 4a. Graphical representation of the total cost with γ

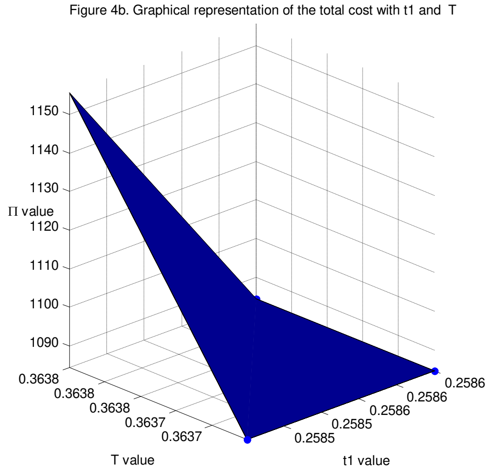


Figure 4b. Graphical representation of the total cost with t1 and T

Figure 5a. Graphical representation of the total cost with A

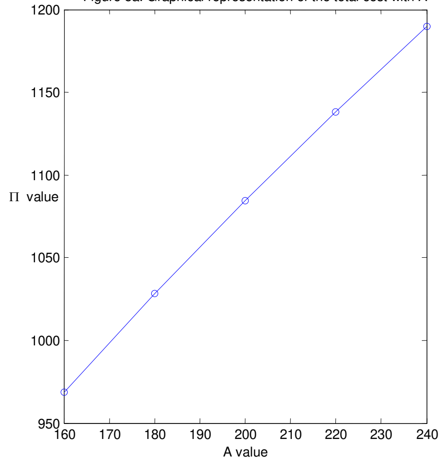


Figure 5b. Graphical representation of the total cost with t1 and T

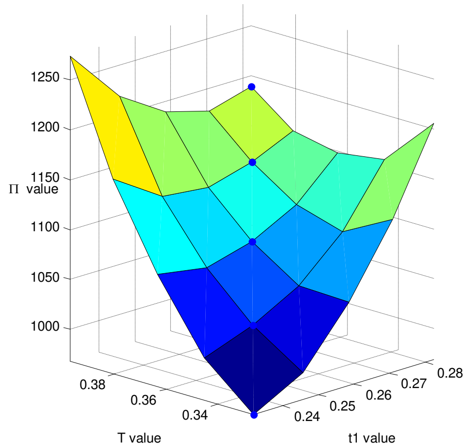


Figure 6a. Graphical representation of the total cost with μ

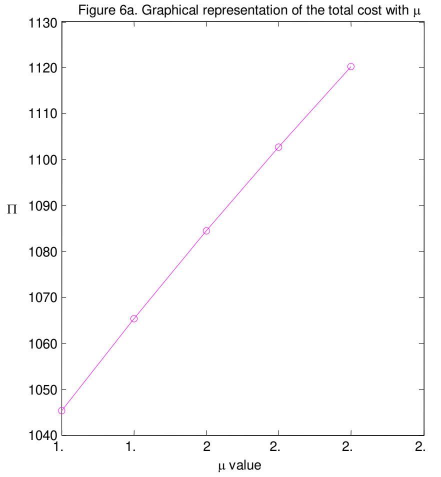


Figure 6b. Graphical representation of the total cost with t1 and T

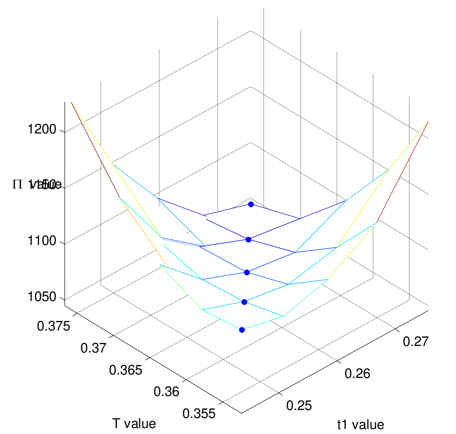


Figure 7a. Graphical representation of the total cost with ν

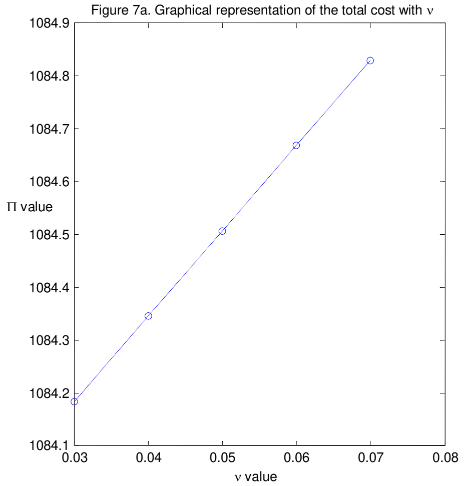


Figure 7b. Graphical representation of the total cost with t1 and T

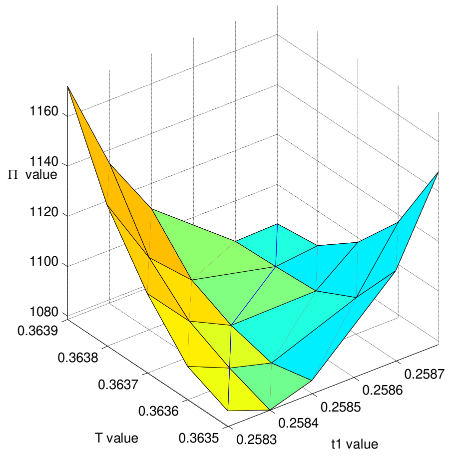


Figure 8a. Graphical representation of the total cost with λ

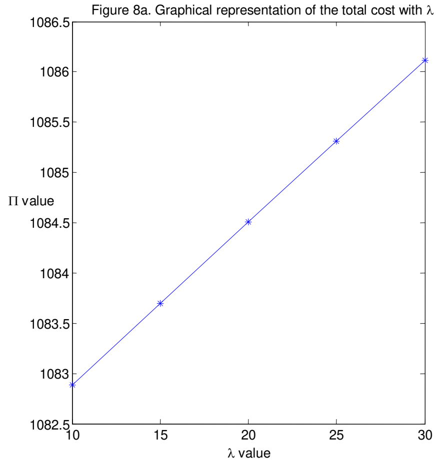


Figure 8b. Graphical representation of the total cost with t1 and T

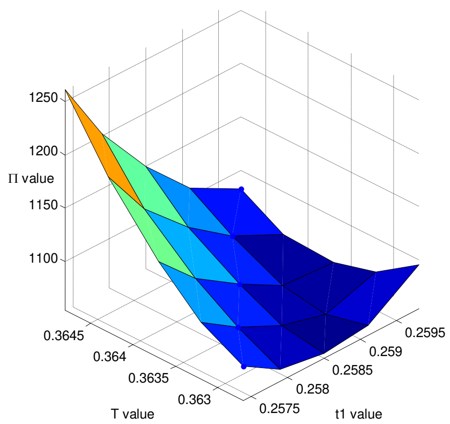


Figure 9a. Graphical representation of the total cost with η

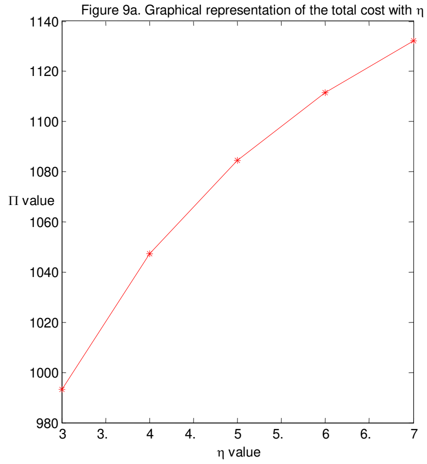


Figure 9b. Graphical representation of the total cost with t1 and T

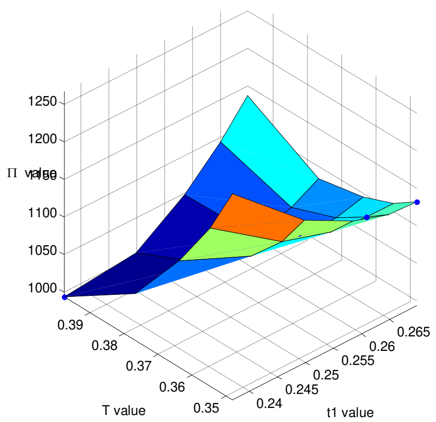


Figure 10a. Graphical representation of the total cost with θ

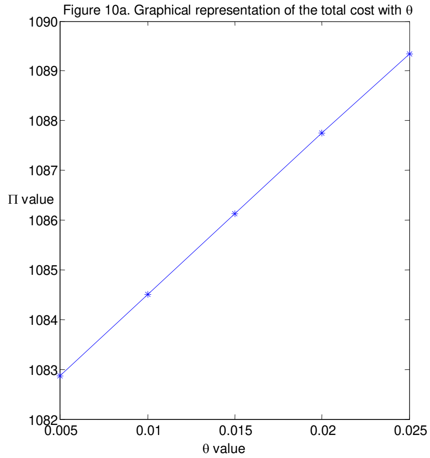
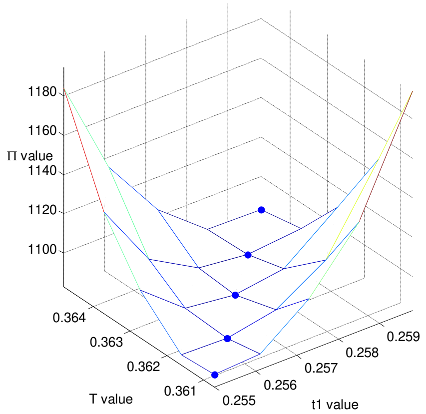


Figure 10b. Graphical representation of the total cost with t1 and T



5.4 Observations

- ❖ If the cost of ordering (A) for each inventory cycle reduces, then the stock in time (t_1), the cycle time of the inventory model (T), Economic order quantity (Q), and the total cost of the inventory system *(Π)* are revealed to be decreasing. It is represented in figures (5a) and (5b). Therefore, the inventory manager is advised to use various successful strategies, such as bringing technical development or finance, to decrease the ordering cost.

- If the holding cost *(μ)* for a unit product increases per unit of time, then the inventory is depleted due to the deterioration and demand of the item *(t_1)*, cycle length *(T)*, and order quantity during the cycle *(Q)* decrease. The inventory total cost *(Π)* increases, as shown in figures (6a) and (6b). This strategy assists the manager in significantly lowering the total carrying cost.
- If the cost per unit of holding reduces due to time variation, then the inventory exhausted time *(t_1)*, length of the cycle *(T)*, and replenishment order quantity *(Q)* grow, and the total inventory cost *(Π)* decreases in figures (7a) and (7b). The inventory manager can use space-reducing strategies to reduce the holding or flexible holding costs.
- If the unit cost of purchase declines, then the value of stock in time *(t_1)*, cycle length *(T)*, and order quantity *(Q)* are susceptible, while the total cost of inventory *(Π)* is less sensitive, which is given in figures (8a) and (8b). The purchase cost is low so the manager can maximize the storage capacity of products in the warehouse.
- If the stock out cost *(η)* increases, then the cycle time of the inventory *(T)* and order quantity *(Q)* decrease, the time when goods run out *(t_1)*, and the inventory cost *(Π)* increases in figures (9a) and (9b). This calls for the manager to opt for buffer stock.
- When the parameters α, β, and θ are raised, then the stock in time *(t_1)*, cycle length *(T)*, and order quantity *(Q)* decrease, and the total cost of the inventory increases, as represented in the figures (2a), (2b), (3a), (3b), (10a) and (10b), respectively.
- When the parameter γ increases, then the inventory exhausted time *(t_1)*, length of the cycle *(T)*, and replenishment order quantity *(Q)* moderately diminish, and the total inventory cost *(Π)* is slightly sensitive, as shown in figures (4a) and (4b).

The inventory model presented in this study offers valuable understanding for retailers aiming to optimize inventory management strategies, especially for products with fluctuating demand dynamics. The sensitivity analysis highlighted the significance of demand parameter 'α', ordering cost, and backlog parameter in influencing retailer expenses. By optimizing these parameters, retailers can effectively manage their inventory costs and improve overall profitability.

A key feature of this model is its consideration of two-phased demand, which accurately reflects real-world scenarios where demand patterns fluctuate over time. This aspect proves particularly useful for managing seasonal product sales, where demand may vary significantly throughout the year. By incorporating two-phased demand, the model provides a more accurate representation of demand dynamics, enabling retailers to adjust their inventory strategies accordingly.

This model's demand parameter 'η' is stable; it allows retailers to prioritize their focus on other key parameters to optimize inventory management and minimize costs effectively, especially during shortage periods.

5.5 Managerial Insides of this model

- The developed mathematical model provides optimal ordering policies considering time-dependent deterioration, linearly varying holding costs, and split demand patterns. Implementing these optimal ordering strategies can lead to significant cost savings for businesses.
- By considering split demand patterns, the model can effectively address seasonal fluctuations in demand. Businesses can adjust their inventory levels and order quantities based on the varying demand patterns throughout the year, thereby reducing excess inventory during low-demand periods and minimizing stockouts during peak seasons.
- The sensitivity analysis in the study highlights the impact of various parameters, such as ordering costs, holding costs, and shortage costs, on the total inventory cost. Managers can use this information to identify cost drivers and implement strategies to minimize holding costs through efficient inventory management practices and to mitigate shortage costs by maintaining optimal inventory levels.
- Incorporating time-dependent deterioration and linear backlog rates into the inventory model enables businesses to manage perishable or deteriorating goods better while addressing backlog situations effectively. This ensures that inventory is utilized efficiently, reducing losses due to product deterioration and minimizing revenue loss from unfulfilled customer demand.
- The sensitivity analysis provides valuable insights into how changes in input parameters affect inventory-related decisions and overall costs. Managers can use this analysis to evaluate the impact of different scenarios and make informed decisions to optimize inventory management practices.
- The computational algorithm developed in the study offers a systematic approach to obtain optimal inventory policies. Businesses can leverage computational tools like MATLAB to implement the algorithm and derive optimal solutions for inventory management challenges.

Overall, the research provides valuable insights and practical recommendations for retailers to optimize their inventory management practices, leading to improved profitability and competitiveness in the market.

6 Conclusions

This research focuses on developing a mathematical model for an inventory system where the demand occurs in a phased manner, split into quadratic and linear demand based on customer behavior and deterioration. This is a unique and innovative attempt to develop an inventory system reflecting modern-day demand patterns.

The developed model considers seasonal demand fluctuations and aims to minimize overall costs and optimize order quantities. The model is exact, and it reflects time-varying markets such as those for seasonal goods whose demand pattern exhibits a nonlinear behavior initially and steadies later on with a linear trend. The demand fluctuation can be accounted for in real life for products like packed drinks, notebooks, air conditioners, etc.

The effectiveness of the Algorithm devised to derive an analytical solution to the EOQ problem is evident from the optimal solutions obtained for an inventory system represented by the numerical parameters depicting reality. The numerical solution is supplemented with the graphical representation of the optimal solution.

Minimizing retailer expenses by understanding cycle time and zero inventory periods is a significant outcome. The research also reveals the emphasis on managing stock-out periods, consumption rates, and the impact of degradation functions in effective inventory planning.

Overall, this research enables the application of the model to real-world scenarios, and the use of sensitivity analysis contributes to the practicality and robustness of the proposed strategy.

Further vistas of research in this direction include incorporating uncertainty by introducing fuzzy-related demand functions, trade credit and quantity discounts.

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