Triangular Decomposition of Tensor Product of Simple Graphs

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Abstract: Let G = (V, E) be a simple connected graph of order p and size q. If $\{G_1, G_2, \ldots, G_n\}$ are edge disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$ then $\{G_1, G_2, \ldots, G_n\}$ is said to be a Decomposition of a graph G. A graph of size $q = \binom{n+2}{3}$ is said to have a Triangular decomposition (TD) if G can be decomposed into n - subgraphs $\{G_1, G_2, \ldots, G_n\}$such that each subgraphs G_i is connected and $|E(G_i)| = \binom{i+1}{2}$ for $1 \leq i \leq n$. In this paper we investigate Triangular decomposition of Tensor product of simple graphs.

Keywords: Triangular decomposition, Tensor product, Wheel Graph.

1. Introduction

A graph G, referred to here is an undirected connected graph without loops or multiple edges. Let $G_1= (V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. The Tensor product $G = G_1 \wedge G_2$ is defined as a graph with vertex set $V_1 x V_2$. Edge set is defined as follows: If $w_1=(u_1,v_1)$ and $w_2= (u_2,v_2)$ are two vertices of G with $u_i \in V_1$ and $v_i \in V_2 (i=1,2)$ then $w_1w_2 \in E(G)$ if and only if $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$. [4] The concept of Continuous Monotonic Decomposition of Graph was introduced by N.Gnana Dhas and J.Paulraj Joseph. [6] S.Asha, and R.kala discussed on Continuous Monotonic Decomposition of some special class of Graphs. Terms not defined here are used in the sense of Harary [1]. In this paper we proved some results on Triangular decomposition of Tensor product of simple graphs.

Definition 1.1 A decomposition of a graph G is a collection of edge disjoint subgraphs $\{G_1, G_2, G_3, \ldots, G_n\}$ of G such that every edge of G belongs to exactly one of the subgraph G_i.

Definition 1.2 A graph G of size $q = \binom{n+2}{3}$ is said to have a Triangular decomposition (TD) if G can be decomposed into n - subgraphs $\{G_1, G_2, G_3, \ldots, G_n\}$ such that each G_i is connected and $|E(G_i)| = \binom{i+1}{2}$ for $1 \le i \le n$.

Definition 1.3. The tensor product $G \times H$ of graphs G and H is a graph such that the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$; and vertices (g, h) and (g', h') are adjacent in
$G \times H$ if and only if g is adjacent to g' in G and h is adjacent to h' in H.

2. Triangular Decomposition of Tensor Productof Simple Graphs

Lemma2.1. If $k = 4s$, then every graph of size $q = \frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_k\}$.

Proof. We have k = 4s, s∈N.We prove this theorem by using induction method. When s =1, k = 4.Then $q = \frac{k(k+1)(k+2)}{6} = \frac{4x5x\,6}{6} = 20$ can be decomposed into $\{G_1, G_2, G_3, G_4\}$. Hence the result is true for s =1.Assume that the result is true for s-1. Then k = 4(s-1) = 4s-4 and $q = \frac{(4s-4)(4s-3)(4s-2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_{4s-4}\}$. Now to prove the result is true for s. Then k = 4s and $q = \frac{4s(4s+1)(4s+2)}{6}$.We have to prove that $\frac{4s(4s+1)(4s+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_{4s}\}$.

$$\text{Now } q = \frac{4s(4s+1)(4s+2)}{6}$$
$$= \frac{64s^3+48\,s^2+8s}{6}$$
$$= \frac{(4s-4)(16s^2-20s+6)}{6} + \left\{\frac{(4s-2)(8s-4)}{2} + \frac{(4s)(8s)}{2}\right\}$$
$$= \frac{(4s-4)(16s^2-8s-12s+6)}{6} + \left\{\frac{(4s-2)(4s-3+4s-1)}{2} + \frac{(4s)(4s-1+4s+1)}{2}\right\}$$
$$= \frac{(4s-4)(4s-3)(4s-2)}{6} + \left\{\frac{(4s-3)(4s-2)}{2} + \frac{(4s-2)(4s-1)}{2} + \frac{(4s-1)(4s)}{2} + \frac{4s(4s+1)}{2}\right\}.$$

Therefore $q = \frac{4s(4s+1)(4s+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_{4s}\}$. Hence by induction hypothesis if $k = 4s$, then every graph of size $q = \frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_k\}$.This completes the proof.

Lemma2.2. If $k + 1 = 4s$, then every graph of size $q = \frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_k\}$.

Proof. We have k = 4s − 1, s ∈ N.We prove this theorem by using induction method. When s =1, k = 3.Then q = $\frac{k(k+1)(k+2)}{6} = \frac{3x4x5}{6}$ = 10 can be decomposed into {G_1,G_2,G_3}. Hence the result is true for s =1.Assume that the result is true for s-1. Then k = 4(s-1)-1 = 4s-5 and q = $\frac{(4s-5)(4s-4)(4s-3)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_{4s-5}}. Now to prove the result is true for s. Then k = 4s-1 and q = $\frac{(4s-1)(4s)(4s+1)}{6}$.We have to prove that q = $\frac{(4s-1)(4s)(4s+1)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_{4s-1}}.

$$\text{Now } q = \frac{(4s-1)(4s)(4s+1)}{6}$$
$$= \frac{64s^3-4s}{6}$$
$$= \frac{(4s-5)(16s^2-28s+12)}{6} + \{\frac{(4s-3)(8s-6)}{2} + \frac{(4s-1)(8s-2)}{2}\}$$
$$= \frac{(4s-5)(16s^2-12s-16s+12)}{6} + \{\frac{(4s-3)(4s-4+4s-2)}{2} + \frac{(4s-1)(4s-2+4s)}{2}\}$$
$$= \frac{(4s-5)(4s-4)(4s-3)}{6} + \{\frac{(4s-4)(4s-3)}{2} + \frac{(4s-3)(4s-2)}{2} + \frac{(4s-2)(4s-1)}{2} + \frac{(4s-1)4s}{2}\}.$$

Therefore q = $\frac{(4s-1)(4s)(4s+1)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_{4s-1}}. Hence by induction hypothesis if k+1= 4s, then every graph of size q = $\frac{k(k+1)(k+2)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_k}.This completes the proof.

Lemma 2.3. If k+2 = 4s, then every graph of size q = $\frac{k(k+1)(k+2)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_k}

Proof. We have k = 4s-2, s ∈ N.We prove this theorem by using induction method. When s=1, k = 2.Then q = $\frac{k(k+1)(k+2)}{6} = \frac{2x3x4}{6}$ = 4 can be decomposed into {G_1, G_2}. Hence the result is true for s =1.Assume that the result is true for s-1. Then k = 4(s-1)-2 = 4s-6 and q = $\frac{(4s-6)(4s-5)(4s-4)}{6}$ can be decomposed into {G_1, G_2, G_3,...,G_{4s-6}}. Now to prove the result is true for s. Then k = 4s-2 and q = $\frac{(4s-2)(4s-1)(4s)}{6}$.We have to prove that q = $\frac{(4s-2)(4s-1)(4s)}{6}$ can be decomposed into {G_1, G_2, G_3,...,G_{4s-2}}.

$$\text{Now } q = \frac{(4s-2)(4s-1)(4s)}{6} = \frac{64s^3-48s^2+8s}{6}$$
$$= \frac{(4s-6)(16s^2-16s-20s+20)}{6} + \{\frac{(4s-4)(8s-8)}{2} + \frac{(4s-2)(8s-4)}{2}\}$$
$$= \frac{(4s-6)(4s-5)(4s-4)}{6} + \{\frac{(4s-5)(4s-4)}{2} + \frac{(4s-4)(4s-3)}{2} + \frac{(4s-3)(4s-2)}{2} + \frac{(4s-2)(4s-1)}{2}\}$$

Therefore q = $\frac{(4s-2)(4s-1)(4s)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_{4s-2}}. Hence by induction hypothesis if k+2 = 4s, then every graph of size q = $\frac{k(k+1)(k+2)}{6}$ can be decomposed into {G_1, G_2, G_3,.........,G_k}.This completes the proof.

Theorem 2.4. Foranyintegerm, the Path graph $P_m \wedge K_2$admitsaTriangular Decomposition $\{G_1, G_2, G_3, \ldots, G_k\}$ifandonlyifthereexistsanintegerksatisfyingthefollowingproperties:

(i) $k = 4r$ or $k = 4r-1$ or $k = 4r-2$, $r \in N$.

(ii) $2m-2 = \frac{k(k+1)(k+2)}{6}$

Proof. Let$G=P_m \wedge K_2$.Then $q(G)= 2m - 2$.AssumeGhasaTriangular Decomposition. By the definition Triangular Decomposition, $q(G) = \frac{k(k+1)(k+2)}{6}$.

Hence $2m-2 = \frac{k(k+1)(k+2)}{6}$.

⇨ $m = \frac{k^3+3k^2+2k+12}{12}$.

Since m is an integer,$k = 4r$ or $k = 4r-1$ or $k = 4r-2$, $r \in N$

Conversely assume (i) $k = 4r$ or $k = 4r-1$ or $k = 4r-2$, $r \in N$. (ii) $2m-2 = \frac{k(k+1)(k+2)}{6}$. Let $G = P_m \wedge K_2$. Then $q(G) = 2m-2$. By lemma 2.1, 2.2 and 2.3, $\frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_k\}$. Thus G admits Triangular Decomposition.

Table 2.5: List of first 10 TD of $P_m \wedge K_2$

m	q(G)	Triangular Decomposition
3	4	G_1, G_2.
6	10	G_1, G_2, G_3.
11	20	G_1, G_2, G_3, G_4
29	56	$G_1, G_2, G_3, \ldots, G_6$
43	84	$G_1, G_2, G_3, \ldots, G_7$
61	120	$G_1, G_2, G_3, \ldots, G_8$
111	220	$G_1, G_2, G_3, \ldots, G_{10}$
144	286	$G_1, G_2, G_3, \ldots, G_{11}$
183	364	$G_1, G_2, G_3, \ldots, G_{12}$
281	560	$G_1, G_2, G_3, \ldots, G_{14}$

Illustration 2.6

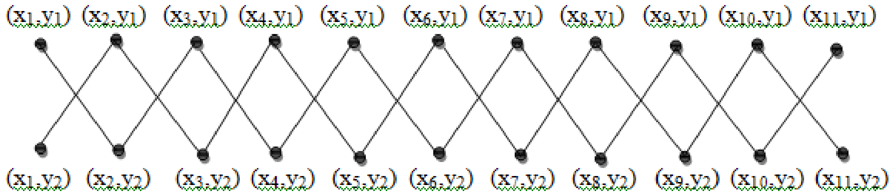

Figure 1: Tensor product of P_{11} with K_2

Triangular Decomposition of $P_{11} \wedge K_2$ as follows

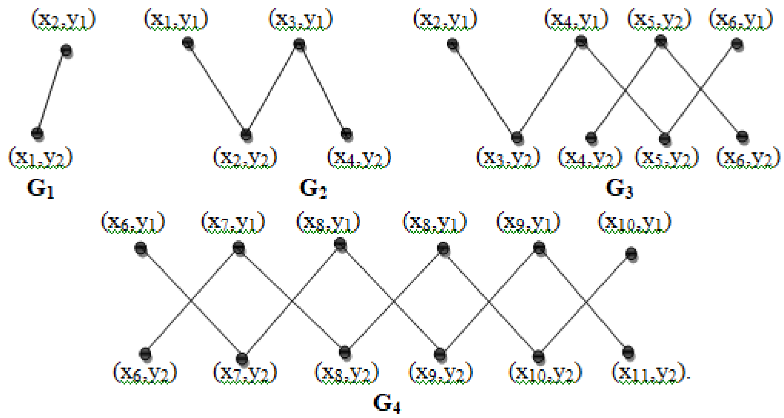

Figure 2: TD $\{G_1, G_2, G_3, G_4\}$ of $P_{11} \wedge K_2$

Lemma2.7 If $k+1=4s$ and $s\equiv 0 (\mathrm{mod}\, 2)$ then every graph of size $q = \frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1,G_2,G_3,\ldots\ldots\ldots,G_k\}$

Proof. We have $k+1 = 4s$ and $s\equiv 0(\mathrm{mod}\,2)$, $s \in N$.We prove this theorem by using induction method. When $s = 2$, $k = 7$.Then $q = \frac{k(k+1)(k+2)}{6} = \frac{7x8x\,9}{6} = 84$ can be decomposed into $\{G_1,G_2,G_3,\ldots\ldots,G_7\}$. Hence the result is true for $s = 2$. Assume that the result is true for $2s-2$. Then $k = 4(2s-2)-1 = 8s-9$ and $q = \frac{(8s-9)(8s-8)(8s-7)}{6}$ can be decomposed into $\{G_1,G_2,G_3,\ldots,G_{8s--9}\}$. Now to prove the result is true for $2s$. Then $k = 8s-1$ and $q = \frac{(8s-1)(8s)(8s+1)}{6}$. We have to prove that $\frac{(8s-1)(8s)(8s+1)}{6}$ can be decomposed into $\{G_1,G_2,G_3,\ldots,G_{8s-1}\}$.

Now $q = \frac{(8s-1)(8s)(8s+1)}{6}$

$= \frac{512s^3-8s}{6}$

$= \frac{(8s-9)(64s^2-120s+56)}{6} + \{\frac{(8s-7)(16s-14)}{2} + \frac{(8s-5)(16s-10)}{2} + \frac{(8s-3)(16s-6)}{2} + \frac{(8s-1)(16s-2)}{2}\}$

$= \frac{(8s-9)(8s-8)(8s-7)}{6} + \{\frac{(8s-8)(8s-7)}{2} + \frac{(8s-7)(8s-6)}{2} + \frac{(8s-6)(8s-5)}{2} + \frac{(8s-5)(8s-4)}{2} + \frac{(8s-4)(8s-3)}{2} + \frac{(8s-3)(8s-2)}{2} + \frac{(8s-2)(8s-1)}{2} + \frac{(8s-1)(8s)}{2}\}$.

Therefore $q = \frac{(8s-1)(8s)(8s+1)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_{8s-1}\}$. Hence by induction hypothesis if $k+1 = 4s$ and $s \equiv 0 (\text{mod } 2)$ then every graph of size $q = \frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_k\}$.This completes the proof.

Theorem 2.8. For any integer m, the wheel graph $W_{m+1} \wedge K_2$ has a Triangular Decomposition $\{G_1, G_2, G_3, \ldots\ldots, G_k\}$ iff there exists an integer k satisfying the following properties.

i) $k = 4r$ or $k = 4r-1$, $r \equiv 0(\text{mod}2)$ or $k = 4r-2$, $r \in N$.

ii) $\frac{k(k+1)(k+2)}{6} = 4m$.

Proof. Let $G = W_{m+1} \wedge K_2$. Then $q(G) = 4m$. Assume $W_{m+1} \wedge K_2$ has a Triangular Decomposition. By the definition of Triangular decomposition, $q(G) = \binom{k+2}{3}$.

Hence $4m = \binom{k+2}{3} = \frac{k(k+1)(k+2)}{6}$.

$\Rightarrow m = \frac{k(k+1)(k+2)}{24}$.

Since m is an integer, $k = 4r$or $k = 4r-1$, $r \equiv 0(\text{mod}2)$ or $k = 4r-2$, $r \in N$.

Conversely assume (i) $k = 4r$or $k = 4r-1$, $r \equiv 0(\text{mod}2)$ or $k = 4r-2$, $r \in N$

(ii) $\frac{k(k+1)(k+2)}{6} = 4m$. Let $G = W_{m+1} \wedge K_2$. Then $q(G) = 4m$. By lemma 2.1, 2.7 and 2.3, $\frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots, G_k\}$. Thus G admits Triangular Decomposition.

Table 2.9: List of first 10, TD of $W_{m+1} \wedge K_2$.

m	q(G)	Triangular Decomposition
5	20	G_1, G_2, G_3, G_4.
14	56	$G_1, G_2, G_3, \ldots\ldots\ldots, G_6$
21	84	$G_1, G_2, G_3, \ldots\ldots\ldots, G_7$
30	120	$G_1, G_2, G_3, \ldots\ldots\ldots, G_8$
55	220	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{10}$
91	364	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{12}$
140	560	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{14}$
170	680	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{15}$
204	816	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{16}$
285	1140	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{18}$

Illustration 2.10

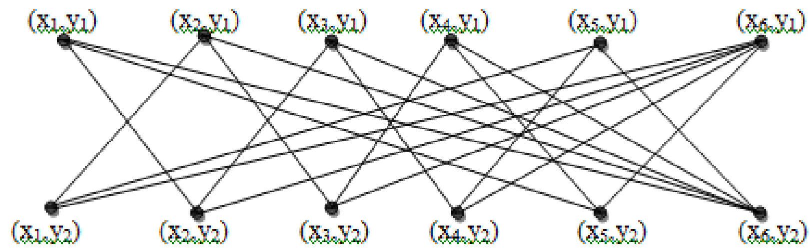

Figure 3: Tensor Product of W_6 with K_2.

Triangular Decomposition of $W_6 \wedge K_2$ as follows.

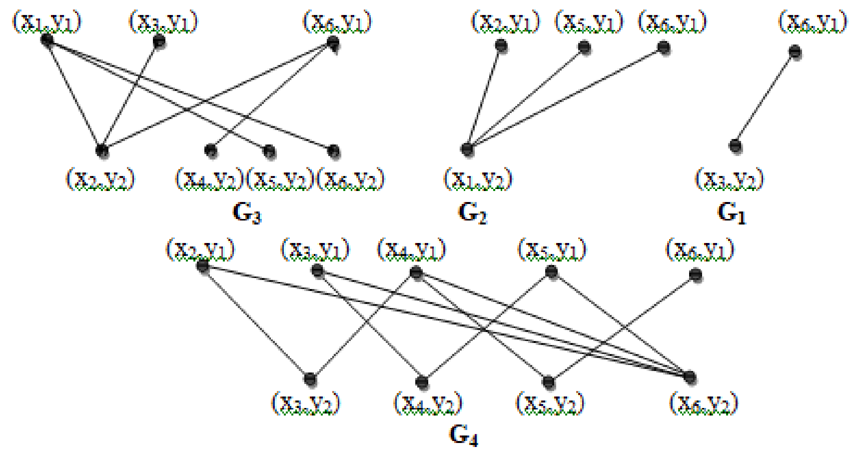

Figure 4: TD $\{G_1, G_2, G_3, G_4\}$ of $W_6 \wedge K_2$.

Theorem 2.11. For any integer m, the cycle graph $C_m \wedge K_2$ has a Triangular Decomposition $\{G_1, G_2, G_3, \ldots, G_k\}$ iff there exists an integer k satisfying the following properties.

i) $k = 4r$ or $k = 4r-1$ or $k = 4r-2$, $r \in N$.

ii) $\frac{k(k+1)(k+2)}{6} = 2m$

Proof. Let $G = C_m \wedge K_2$. Then $q(G) = 2m$. Assume $C_m \wedge K_2$ has a Triangular Decomposition $\{G_1, G_2, G_3, \ldots, G_k\}$. By the definition of Triangular Decomposition, $q(G) = \binom{k+2}{3}$.

Hence $2m = \binom{k+2}{3} = \frac{k(k+1)(k+2)}{6}$.

$\Rightarrow \quad m = \frac{k(k+1)(k+2)}{12}$.

Since m is an integer, $k = 4r$ or $k = 4r-1$ or $k = 4r-2$, $r \in N$.

Conversely assume (i) $k = 4r$ or $k = 4r-1$ or $k = 4r-2$, $r \in N$ (ii) $\frac{k(k+1)(k+2)}{6} = 2m$.

Let $G = C_m \wedge K_2$. Then $q(G) = 2m$. By lemma 2.1, 2.2 and 2.3, $\frac{k(k+1)(k+2)}{6}$ can be decomposed into $\{G_1, G_2, G_3, \ldots\ldots\ldots, G_k\}$. Thus G admits Triangular Decomposition.

Table 2.12: List of first 10 TD of $C_m \wedge K_2$

m	q(G)	Triangular Decomposition
5	10	G_1, G_2, G_3
10	20	G_1, G_2, G_3, G_4
28	56	$G_1, G_2, G_3, \ldots\ldots\ldots, G_6$
42	84	$G_1, G_2, G_3, \ldots\ldots\ldots, G_7$
60	120	$G_1, G_2, G_3, \ldots\ldots\ldots, G_8$
110	220	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{10}$
143	286	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{11}$
182	364	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{12}$
280	560	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{14}$
340	680	$G_1, G_2, G_3, \ldots\ldots\ldots, G_{15}$

Illustration 2.13

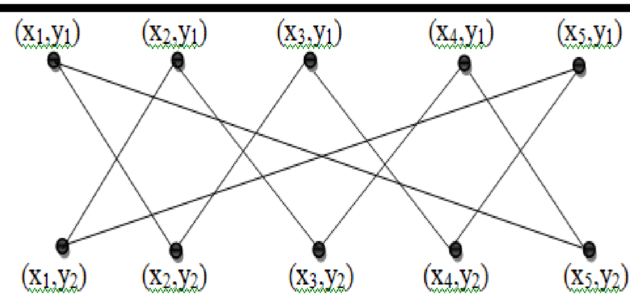

Figure 5: Tensor Product of C_5 with K_2

Triangular Decomposition of $C_5 \wedge K_2$ as follows

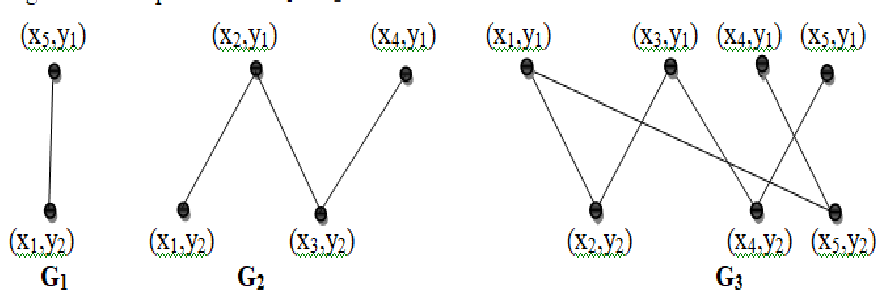

Figure 6: TD $\{G_1, G_2, G_3\}$ of $C_5 \wedge K_2$

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